The index growth and multiplicity of closed geodesics

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Abstract

In the recent paper [LoD1], we classified closed geodesics on Finsler manifolds into rational and irrational two families, and gave a complete understanding on the index growth properties of iterates of rational closed geodesics. This study yields that a rational closed geodesic can not be the only closed geodesic on every irreversible or reversible (including Riemannian) Finsler sphere, and that there exist at least two distinct closed geodesics on every compact simply connected irreversible or reversible (including Riemannian) Finsler 3*-dimensional manifold. In this paper, we study the index growth properties of irrational closed geodesics on Finsler manifolds. This study allows us to extend results in [LoD1] on rational and in [DuL1], [Rad4] and [Rad5] on completely non-degenerate closed geodesics on spheres and* $\mathbf{CP}^2$ *to every compact simply connected Finsler manifold. Then we prove the existence of at least two distinct closed geodesics on every compact simply connected irreversible or reversible (including Riemannian) Finsler* 4*-dimensional manifold.*

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1 Introduction and main results

It has been a long-standing problem in dynamical systems and differential geometry whether every compact Riemannian manifold has infinitely many distinct closed geodesics. D. Gromoll and W. Meyer [GrM1] in 1969 proved the following result:

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Theorem A. (cf. [GrM1]) *On a compact Riemannian manifold there exist infinitely many closed geodesics, if the free loop space of this manifold has an unbounded sequence of Betti numbers.*

Stimulated by this result, M. Vigué-Poirrier and D. Sullivan [ViS1] in 1976 established following result:

Theorem B. (cf. [ViS1]) *The free loop space of a compact simply connected Riemannian manifold M has no unbounded sequence of Betti numbers if and only if the rational cohomology algebra of M possess only one generator.*

Both of the two theorems were generalized to corresponding Finsler manifolds by H. Matthias in 1980 (cf. [Mat1]). Therefore based on these two theorems, the most interesting manifolds in this multiplicity problem are those compact simply connected manifolds satisfying

$$H^*(M;\mathbf{Q}) \cong T_{d,h+1}(x) = \mathbf{Q}[x]/(x^{h+1} = 0) \tag{1.1}$$

with a generator x of degree $d \geq 2$ and hight $h+1 \geq 2$. The main examples are the compact rank one symmetric spaces, i.e., spheres S^d of dimension d with $h = 1$, complex projective spaces $\mathbf{C}P^h$ of dimension $2h$ with $d = 2$, quaternionic projective spaces $\mathbf{H}P^h$ of dimension $4h$ with $d = 4$, and the Cayley plane $\mathbf{C}aP^2$ of dimension 16 with $d = 8$ and $h = 2$.

The studies of closed geodesics on such manifolds can be chased back to J. Jacobi, J. Hadamard, H. Poincaré, G. D. Birkhoff, M. Morse, L. Lyusternik and Schnirelmann and others. Specially G. D. Birkhoff established the existence of at least one closed geodesic on every Riemannian sphere S^d with $d \geq 2$ (cf. [Bir1]). Later L. Lyusternik and A. Fet proved the existence of at least one closed geodesic on every compact Riemannian manifold (cf. [LyF1]). An important breakthrough on this problem is due to V. Bangert [Ban2] and J. Franks [Fra1] around 1990, who proved that there exist always infinitely many closed geodesics on every Riemannian 2-sphere (cf. also [Hin1] and [Hin2]). But when the dimension of a compact simply connected manifold is greater than 2, we are not aware of any multiplicity results on the existence of at least two closed geodesics without pinching or bumpy conditions even on spheres (cf. [Ano1], [Ban1], [Kli1], [BTZ1], [BTZ2] and [DuL1], [Rad4], [Rad5]), except the Theorem C below proved recently in [LoD1].

When one considers irreversible Finsler metrics, the problem of counting closed geodesics becomes more delicate because of A. Katok's famous example of 1973 which shows that there exist some irreversible Finsler metrics on S^d with only finitely many closed geodesics (cf. [Kat1] and [Zil2]). In [HWZ1] of 2003, H. Hofer, K. Wysocki and E. Zehnder proved that there exist either two or infinitely many distinct prime closed geodesics on a Finsler (S^2, F) provided that all the iterates of all closed geodesics are non-degenerate and the stable and unstable manifolds of all hyperbolic closed geodesics intersect transversally. In [BaL1] of 2005, V. Bangert and Y. Long proved that on every irreversible Finsler S^2 there always exist at least two distinct prime closed geodesics.

Note that in the recent [LoD1], we have proved the following

Theorem C. *There exist always at least two distinct prime (geometrically distinct) closed geodesics for every irreversible (or reversible, specially Riemannian) Finsler metric on any 3-dimensional compact simply connected manifold, where the typical case is S^3.*

To further our study on the multiplicity of closed geodesics, we note that in the famous book [Mor1] of 1934, M. Morse studied closed geodesics on ellipsoids. Specially he proved that for any given integer $N > 0$, every closed geodesic c of a d-dimensional ellipsoid E^d in $\mathbf{R}^{d+1}$ which is not an iterate of some main ellipse must have Morse index $i(c) \geq N$, provided all the semi-axes of E^d are less than 1 and sufficiently closed to 1. Consequently the Betti numbers at dimensions less than N of the free loop space of such an E^d can be generated by iterates of the $d+1$ main ellipses on E^d only. His this result suggests that it is necessary to study asymptotic and growth properties of Morse indices of iterates of prime closed geodesics on the manifold in order to get multiplicity results.

In the recent paper [LoD1], we classified prime closed geodesics on any compact Finsler manifold M into two families: rational and irrational. Here a prime closed geodesic is **rational**, if its basic normal form decomposition (cf. Section 3 below) introduced by Y. Long in [Lon1] and [Lon2] contains no 2×2 rotation matrix $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $\theta/\pi \in \mathbf{R} \setminus \mathbf{Q}$, and **irrational** otherwise. A prime closed geodesic is **completely non-degenerate**, if all of its iterates c^m are non-degenerate.

Recall that on a compact Finsler manifold (M, F), a closed geodesic $c : S^1 = \mathbf{R}/\mathbf{Z} \to M$ is **prime**, if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the m-th iteration c^m of c is defined by $c^m(t) = c(mt)$ for $m \in \mathbf{N}$. The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1-t)$ for $t \in S^1$. Two prime closed geodesics c_1 and c_2 on a Finsler manifold (M, F) (or Riemannian manifold (M, g)) are **distinct** (or **geometrically distinct**), if they do not differ by an S^1-action (or $O(2)$-action).

In [LoD1], the index growth properties of rational closed geodesics are completely understood. This result is used to prove that on every (irreversible or reversible) Finsler sphere S^d, it is impossible that there exists only one prime closed geodesic which is rational.

In the Section 3 of this paper, we study first the growth properties of indices of iterates of irrational closed geodesics. We show that if the initial index of a prime closed geodesic is not too small, then the Morse indices $i(c^m)$ is monotone in $m \geq 1$. When this index monotonicity does not hold, we prove that for a closed geodesic c, there exist infinitely many positive integers T such that the indices $\{i(c^m)\}_{m>T}$ and the indices $\{i(c^m)\}_{m<T}$ are suitably separated by the sum of $i(c^T)$ and some constant (see Theorem 3.21 below). We call this property the **quasi-monotonicity**.

As applications of these studies, in Section 4 we then generalize the result in [LoD1] on rational closed geodesics on spheres, and the results in [DuL1], [Rad4], and [Rad5] on completely non-degenerate closed geodesics on spheres and $\mathbf{C}P^2$ to all compact simply connected manifolds. That is:

Theorem 1.1. *For every irreversible (or reversible, specially Riemannian) Finsler metric F on any compact simply connected manifold, if there exists only one prime (geometrically distinct) closed geodesic, it can be neither rational nor completely non-degenerate.*

Then using above results we study the 4-dimensional case in the Sections 5 and 6 respectively, and prove the following theorems.

Theorem 1.2. *For every irreversible Finsler metric F on any compact simply connected 4-dimensional manifold, there always exist at least two distinct prime closed geodesics.*

Theorem 1.3. *For every reversible Finsler metric F on any compact simply connected 4-dimensional manifold, there always exist at least two geometrically distinct closed geodesics. In particular, it holds for every such Riemannian manifold.*

For reader's conveniences, in Section 2 we briefly review some known results on closed geodesics, and compute the precise sums of Betti numbers of the S^1-invariant free loop space of compact simply connected manifolds satisfying the condition (1.1).

In this paper, we denote by $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ the sets of positive integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers respectively. We define the functions $[a] = \max\{k \in \mathbf{Z} \,|\, k \le a\}$, $\{a\} = a - [a]$, $E(a) = \min\{k \in \mathbf{Z} \,|\, k \ge a\}$ and $\varphi(a) = E(a) - [a]$. Denote by $^{\#}A$ the number of elements in a finite set A. In this paper, we use only singular homology modules with $\mathbf{Q}$-coefficients.

2 Critical point theory of closed geodesics

2.1 Critical modules for closed geodesics

Let M be a compact and simply connected manifold with a Finsler metric F. Closed geodesics are critical points of the energy functional $E(\gamma) = \frac{1}{2}\int_{S^1} F(\gamma(t), \dot{\gamma}(t))^2 dt$ on the Hilbert manifold ΛM of H^1-maps from S^1 to M. An S^1-action is defined by $(s\cdot\gamma)(t) = \gamma(t+s)$ for all $\gamma \in \Lambda M$ and $s, t \in S^1$. The index form of the functional E is well defined along any closed geodesic c on M, which we denote by $E''(c)$. As usual, denote by $i(c)$ and $\nu(c)$ the Morse index and nullity of E at c. For a closed geodesic c, denote by c^m the m-fold iteration of c and $\Lambda(c^m) = \{\gamma \in \Lambda M \,|\, E(\gamma) < E(c^m)\}$. Recall that respectively the *mean index* $\hat{i}(c)$ and the S^1-*critical modules* of c^m are defined by

$$\hat{i}(c) = \lim_{m\to\infty} \frac{i(c^m)}{m}, \quad \overline{C}_*(E, c^m) = H_*\left((\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1\right). \tag{2.1}$$

If c has multiplicity m, then the subgroup $\mathbf{Z}_m = \{\frac{n}{m} : 0 \le n < m\}$ of S^1 acts on $\overline{C}_k(E, c)$. As on p.59 of [Rad2], for $m \ge 1$, let $H_*(X, A)^{\pm \mathbf{Z}_m} = \{[\xi] \in H_*(X, A) : T_*[\xi] = \pm\xi\}$, where T is a generator of the $\mathbf{Z}_m$ action. On S^1-critical modules of c^m, the following lemma holds:

Lemma 2.1. (cf. Satz 6.11 of [Rad2]) *Suppose c is a prime closed geodesic on a compact Finsler manifold M. Then there exist two sets $U_{c^m}^-$ and N_{c^m}, the so-called local negative disk and*

the local characteristic manifold at c^m respectively, such that $\nu(c^m) = \dim N_{c^m}$ and

$$\begin{aligned}\overline{C}_q(E, c^m) &\equiv H_q\left((\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1\right) \\ &= \left(H_{i(c^m)}(U^-_{c^m} \cup \{c^m\}, U^-_{c^m}) \otimes H_{q-i(c^m)}(N^-_{c^m} \cup \{c^m\}, N^-_{c^m})\right)^{+\mathbf{Z}_m},\end{aligned}$$

(i) When $\nu(c^m) = 0$, there holds

$$\overline{C}_q(E, c^m) = \begin{cases} \mathbf{Q}, & \text{if } \ i(c^m) = i(c)\,(\mathrm{mod}2) \ \text{ and } \ q = i(c^m), \\ 0, & \text{otherwise}, \end{cases}$$

(ii) When $\nu(c^m) > 0$, let $\epsilon(c^m) = (-1)^{i(c^m)-i(c)}$, then there holds

$$\overline{C}_q(E, c^m) = H_{q-i(c^m)}(N^-_{c^m} \cup \{c^m\}, N^-_{c^m})^{\epsilon(c^m)\mathbf{Z}_m}.$$

Let

$$k_j(c^m) \equiv \dim\, H_j(N^-_{c^m} \cup \{c^m\}, N^-_{c^m}), \quad k_j^{\pm 1}(c^m) \equiv \dim\, H_j(N^-_{c^m} \cup \{c^m\}, N^-_{c^m})^{\pm\mathbf{Z}_m}. \tag{2.2}$$

Then we have

Lemma 2.2. (cf. [Rad2], [BaL1], [DuL1]) *Let c be a closed geodesic on a Finsler manifold M.*

(i) There hold $0 \le k_j^{\pm 1}(c^m) \le k_j(c^m)$ for $m \ge 1$ and $j \in \mathbf{Z}$, $k_j(c^m) = 0$ whenever $j \notin [0, \nu(c^m)]$ and $k_0(c^m) + k_{\nu(c^m)}(c^m) \le 1$. If $k_0(c^m) + k_{\nu(c^m)}(c^m) = 1$, then $k_j(c^m) = 0$ when $j \in (0, \nu(c^m))$.

(ii) For any $m \in \mathbf{N}$, there hold $k_0^{+1}(c^m) = k_0(c^m)$ and $k_0^{-1}(c^m) = 0$. In particular, if c^m is non-degenerate, there hold $k_0^{+1}(c^m) = k_0(c^m) = 1$, and $k_0^{-1}(c^m) = k_j^{\pm 1}(c^m) = 0$ for all $j \ne 0$.

(iii) Suppose for some integer $m = np \ge 2$ with n and $p \in \mathbf{N}$ the nullities satisfy $\nu(c^m) = \nu(c^n)$. Then there hold $k_j(c^m) = k_j(c^n)$ and $k_j^{\pm 1}(c^m) = k_j^{\pm 1}(c^n)$ for any integer j.

2.2 Rademacher-type mean index identity for closed geodesics

Let (M, F) be a compact and simply connected Finsler manifold with finitely many prime closed geodesics. It is well known that for every prime closed geodesic c on (M, F), there holds either $\hat{i}(c) > 0$ and then $i(c^m) \to +\infty$ as $m \to +\infty$, or $\hat{i}(c) = 0$ and then $i(c^m) = 0$ for all $m \in \mathbf{N}$. Denote those prime closed geodesics on (M, F) with positive mean indices by $\{c_j\}_{1 \le j \le k}$. In [Rad1] and [Rad2], Rademacher established a celebrated mean index identity relating all the c_js with the global homology of M (cf. Section 7, specially Satz 7.9 of [Rad2]) for compact simply connected Finsler manifolds.

For each $m \in \mathbf{N}$, let $\epsilon = \epsilon(c^m) = (-1)^{i(c^m)-i(c)}$ and

$$\begin{aligned}K(c^m) &\equiv (k_0^{\epsilon}(c^m), k_1^{\epsilon}(c^m), \dots, k_{2\dim M-2}^{\epsilon}(c^m)) \\ &= (k_0^{\epsilon(c^m)}(c^m), k_1^{\epsilon(c^m)}(c^m), \dots, k_{\nu(c^m)}^{\epsilon(c^m)}(c^m), 0, \dots, 0).\end{aligned} \tag{2.3}$$

Lemma 2.3. (cf. Lemmas 7.1 and 7.2 of [Rad2], cf. also [LoD1]) *Let c be a prime orientable closed geodesic on a compact Finsler manifold (M,F). Then there exist a minimal integer $N = N(c) \in \mathbf{N}$ such that $\nu(c^{m+N}) = \nu(c^m)$, $i(c^{m+N}) - i(c^m) \in 2\mathbf{Z}$, and $K(c^{m+N}) = K(c^m)$, $\forall\, m \in \mathbf{N}$.*

Lemma 2.4. (Satz 7.9 of [Rad2], cf. also [LoD1]) *Let (M,F) be a compact simply connected Finsler manifold with $H^*(M,\mathbf{Q}) = T_{d,h+1}(x)$. Denote prime closed geodesics on (M,F) with positive mean indices by $\{c_j\}_{1\le j\le k}$ for some $k \in \mathbf{N}$. Then the following identity holds*

$$\sum_{j=1}^{k} \frac{\hat{\chi}(c_j)}{\hat{i}(c_j)} = B(d,h) = \begin{cases} -\frac{h(h+1)d}{2d(h+1)-4}, & d\,\text{even}, \\ \frac{d+1}{2d-2}, & d\,\text{odd}, \end{cases} \tag{2.4}$$

where $\dim M = hd$, *$h = 1$ when M is a sphere S^d of dimension d and*

$$\hat{\chi}(c) = \frac{1}{N(c)} \sum_{\substack{0\le l_m \le \nu(c^m) \\ 1\le m\le N(c)}} (-1)^{i(c^m)+l_m} k_{l_m}^{\epsilon(c^m)}(c^m) \;\in\; \mathbf{Q}. \tag{2.5}$$

2.3 The structure of $H_*(\Lambda M/S^1, \Lambda^0 M/S^1; \mathbf{Q})$

Set $\overline{\Lambda}^0 = \overline{\Lambda}^0 M = \{\text{constant point curves in } M\} \cong M$. Let (X,Y) be a space pair such that the Betti numbers $b_i = b_i(X,Y) = \dim H_i(X,Y;\mathbf{Q})$ are finite for all $i \in \mathbf{Z}$. As usual the *Poincaré series* of (X,Y) is defined by the formal power series $P(X,Y) = \sum_{i=0}^{\infty} b_i t^i$. We need the following well known version of results on Betti numbers.

Lemma 2.5. (cf. Theorem 2.4 and Remark 2.5 of [Rad1], cf. also Proposition 2.4 of [LoD1]) *Let (S^d, F) be a d-dimensional Finsler sphere.*

(i) *When d is odd, the Betti numbers are given by*

$$\begin{aligned} b_j &= \operatorname{rank} H_j(\Lambda S^d/S^1, \Lambda^0 S^d/S^1; \mathbf{Q}) \\ &= \begin{cases} 2, & \text{if} \quad j \in \mathcal{K} \equiv \{k(d-1) \,|\, 2 \le k \in \mathbf{N}\}, \\ 1, & \text{if} \quad j \in \{d-1+2k \,|\, k \in \mathbf{N}_0\} \setminus \mathcal{K}, \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{2.6}$$

For any $k \in \mathbf{N}$ and $k \ge d-1$, there holds

$$\begin{aligned} \sum_{j=0}^{k} (-1)^j b_j &= \sum_{0\le 2j\le k} b_{2j} \\ &= [\frac{k}{d-1}] + [\frac{k}{2}] - \frac{d-1}{2} \\ &= \frac{k(d+1)}{2(d-1)} - \frac{d-1}{2} - \epsilon_{d,1}(k) \\ &\le \frac{k(d+1)}{2(d-1)} - \frac{d-1}{2}. \end{aligned} \tag{2.7}$$

where $\epsilon_{d,1}(k) = \{\frac{k}{d-1}\} + \{\frac{k}{2}\} \in [0, \frac{3}{2} - \frac{1}{2(d-1)})$.

(ii) *When d is even, the Betti numbers are given by*

$$b_j = \mathrm{rank} H_j(\Lambda S^d/S^1, \Lambda^0 S^d/S^1; \mathbf{Q}) = \begin{cases} 2, & \textit{if} \quad j \in \mathcal{K} \equiv \{k(d-1) \,|\, 3 \le k \in (2\mathbf{N}+1)\}, \\ 1, & \textit{if} \quad j \in \{d-1+2k \,|\, k \in \mathbf{N}_0\} \setminus \mathcal{K}, \\ 0 & \textit{otherwise}. \end{cases} \tag{2.8}$$

For any $k \in \mathbf{N}$ and $k \ge d-1$, there holds

$$\begin{aligned} -\sum_{j=0}^{k}(-1)^j b_j &= \sum_{0\le 2j-1\le k} b_{2j-1} \\ &= \left[\frac{[\frac{k}{d-1}]+1}{2}\right] + [\frac{k+1}{2}] - \frac{d}{2} \\ &= \frac{kd}{2(d-1)} - \frac{d-2}{2} - \{\frac{[\frac{k}{d-1}]+1}{2}\} - \{\frac{k+1}{2}\} - \frac{1}{2}\{\frac{k}{d-1}\} \\ &\le \frac{kd}{2(d-1)} - \frac{d-2}{2}. \end{aligned} \tag{2.9}$$

Proof. It suffices to prove (2.7) and (2.9).

When d is odd, for any $k \in \mathbf{N}$ and $m \in [0, d-1)$, we have

$$\begin{aligned} \sum_{0\le j\le k(d-1)+m} b_j &= \sum_{0\le 2j\le k(d-1)+m} b_{2j} \\ &= 2(k-1) + \frac{k(d-1)-(d-3)}{2} - (k-1) + [\frac{m}{2}] \\ &= k + \frac{(k-1)(d-1)}{2} + [\frac{m}{2}]. \end{aligned}$$

Thus for any integer $k \ge d-1$, because d is odd, we obtain

$$\begin{aligned} \sum_{0\le 2j\le k} b_{2j} &= [\frac{k}{d-1}] + \frac{([\frac{k}{d-1}]-1)(d-1)}{2} + \left[\frac{k-[\frac{k}{d-1}](d-1)}{2}\right] \\ &= [\frac{k}{d-1}] + [\frac{k}{2}] - \frac{d-1}{2} \\ &= \frac{k(d+1)}{2(d-1)} - \frac{d-1}{2} - \{\frac{k}{d-1}\} - \{\frac{k}{2}\} \\ &\le \frac{k(d+1)}{2(d-1)} - \frac{d-1}{2}. \end{aligned}$$

This proves (2.7).

When d is even, for any odd $k \in \mathbf{N}$ and $m \in [0, 2(d-1))$, we have

$$\begin{aligned} \sum_{0\le j\le k(d-1)+m} b_j &= \sum_{0\le 2j-1\le k(d-1)+m} b_{2j-1} \\ &= 2\,\frac{k-1}{2} + \frac{k(d-1)-(d-3)}{2} - \frac{k-1}{2} + [\frac{m}{2}] \\ &= \frac{k+1}{2} + \frac{(k-1)(d-1)}{2} + [\frac{m}{2}]. \end{aligned}$$

Note that for an integer $l > 0$ there holds

$$2\left[\frac{l+1}{2}\right] - 1 = \begin{cases} l, & \text{for } l \in 2\mathbf{N}-1, \\ l-1, & \text{for } l \in 2\mathbf{N}. \end{cases}$$

Thus for any integer $k \geq d-1$, because d is even, we obtain

$$\begin{aligned}
\sum_{0\leq j\leq k} b_j &= \sum_{0\leq 2j-1\leq k} b_{2j-1} \\
&= \frac{1}{2}\left((2\left[\frac{[\frac{k}{d-1}]+1}{2}\right]-1)+1\right) + \frac{1}{2}\left((2\left[\frac{[\frac{k}{d-1}]+1}{2}\right]-1)-1\right)(d-1) \\
&\qquad + \left[\frac{1}{2}\left(k-(2\left[\frac{[\frac{k}{d-1}]+1}{2}\right]-1)(d-1)\right)\right] \\
&= \left[\frac{[\frac{k}{d-1}]+1}{2}\right] + \left(\left[\frac{[\frac{k}{d-1}]+1}{2}\right]-1\right)(d-1) \\
&\qquad + \left[\frac{k+(d-1)}{2}\right] - \left[\frac{[\frac{k}{d-1}]+1}{2}\right](d-1) \\
&= \left[\frac{[\frac{k}{d-1}]+1}{2}\right] + \left[\frac{k+1}{2}\right] - \frac{d}{2} \\
&= \frac{kd}{2(d-1)} - \frac{d-2}{2} - \{\frac{[\frac{k}{d-1}]+1}{2}\} - \{\frac{k+1}{2}\} - \frac{1}{2}\{\frac{k}{d-1}\} \\
&\leq \frac{kd}{2(d-1)} - \frac{d-2}{2}.
\end{aligned}$$

This proves (2.9). ∎

For a compact and simply connected Finsler manifold M with $H^*(M;\mathbf{Q}) \cong T_{d,h+1}(x)$, when d is odd, then $x^2 = 0$ and $h = 1$ in $T_{d,h+1}(x)$. Thus M is rationally homotopy equivalent to S^d (cf. Remark 2.5 of [Rad1] and [Hin1]). Therefore, next we only consider the case when d is even.

Then we have the following result.

Lemma 2.6. (cf. Theorem 2.4 of [Rad1]) *Let M be a compact simply connected manifold with $H^*(M;\mathbf{Q}) \cong T_{d,h+1}(x)$ for some integer $h \geq 2$ and even integer $d \geq 2$. Let $D = d(h+1)-2$ and*

$$\begin{aligned}\Omega(d,h) = \{k \in 2\mathbf{N}-1 \quad | \quad & iD \leq k-(d-1) = iD+jd \leq iD+(h-1)d \\ & \textit{for some } i \in \mathbf{N} \textit{ and } j \in [1,h-1]\}.\end{aligned} \tag{2.10}$$

Then the Betti numbers of the free loop space of M defined by $b_q = \mathrm{rank} H_q(\Lambda M/S^1, \Lambda^0 M/S^1;\mathbf{Q})$ for $q \in \mathbf{Z}$ are given by

$$b_q = \begin{cases} 0, & \textit{if } q \textit{ is even or } q \leq d-2, \\ [\frac{q-(d-1)}{d}]+1, & \textit{if } q \in 2\mathbf{N}-1 \textit{ and } d-1 \leq q < d-1+(h-1)d, \\ h+1, & \textit{if } q \in \Omega(d,h), \\ h, & \textit{otherwise}. \end{cases} \tag{2.11}$$

For every integer $k \geq d-1+(h-1)d = hd-1$, we have

$$\begin{aligned}
\sum_{q=0}^{k} b_q &= \frac{h(h+1)d}{2D}(k-(d-1)) - \frac{h(h-1)d}{4} + 1 + \epsilon_{d,h}(k) \\
&\leq h(\frac{D}{2}+1)\frac{k-(d-1)}{D} - \frac{h(h-1)d}{4} + 1 + \{\frac{D}{hd}\{\frac{k-(d-1)}{D}\}\} \\
&< h(\frac{D}{2}+1)\frac{k-(d-1)}{D} - \frac{h(h-1)d}{4} + 2, \qquad (2.12)
\end{aligned}$$

where

$$\begin{aligned}
\epsilon_{d,h}(k) &= \{\frac{D}{hd}\{\frac{k-(d-1)}{D}\}\} - (\frac{2}{d}+\frac{d-2}{hd})\{\frac{k-(d-1)}{D}\} \\
&\quad -h\{\frac{D}{2}\{\frac{k-(d-1)}{D}\}\} - \{\frac{D}{d}\{\frac{k-(d-1)}{D}\}\}, \qquad (2.13)
\end{aligned}$$

and there hold $\epsilon_{d,h}(k) \in (-(h+2),1)$ and $\epsilon_{d,1}(k) \in (-2,0]$ for all integer $k \geq d-1$.

Proof. For a compact and simply connected Finsler manifold M with $H^*(M;\mathbf{Q}) \cong T_{d,h+1}(x)$ and some even integer d, the following Poincaré series was computed out by Theorem 2.4 of [Rad1]

$$\sum_{q=0}^{+\infty} b_q\, t^q \equiv P(\Lambda M/S^1, \Lambda^0 M/S^1)(t) = t^{d-1}\left(\frac{1}{1-t^2} + \frac{t^{d(h+1)-2}}{1-t^{d(h+1)-2}}\right)\frac{1-t^{dh}}{1-t^d}. \qquad (2.14)$$

Thus we get

$$\begin{aligned}
\sum_{q=0}^{+\infty} b_q\, t^q &= t^{d-1}\left(\sum_{i=0}^{+\infty} t^{2i} + \sum_{i=1}^{+\infty} t^{iD}\right)\sum_{j=0}^{h-1} t^{jd} \\
&= t^{d-1}\left(\sum_{j=0}^{h-1}\sum_{i=0}^{+\infty} t^{2i+jd} + \sum_{j=0}^{h-1}\sum_{i=1}^{+\infty} t^{iD+jd}\right). \qquad (2.15)
\end{aligned}$$

For the first sum in (2.15), we have

$$\begin{aligned}
\sum_{k\in\mathbf{Z}} u_k t^k &\equiv \sum_{j=0}^{h-1}\sum_{i=0}^{+\infty} t^{2i+jd} \\
&= \sum_{j=0}^{h-2}(j+1)\sum_{2i=jd}^{(j+1)d-2} t^{2i} + h\sum_{i=0}^{+\infty} t^{(h-1)d+2i}, \qquad (2.16)
\end{aligned}$$

where (2.16) is obtained by listing all items t^{2i+jd} into a strip with j running from 0 to $h-1$ downwards and i running from 0 to $+\infty$ rightwards, and then summing up all terms with the exponents 0, 2, ..., $d-2$, d, ..., $(h-1)d-2$, $(h-1)d$, ..., respectively. Therefore we obtain

$$u_k = \begin{cases} 0, & \text{if } k \in 2\mathbf{Z}-1 \text{ or } k<0, \\ [\frac{k}{d}]+1, & \text{if } k \in 2\mathbf{N}_0 \text{ and } 0 \leq k < (h-1)d, \\ h, & \text{if } k \in 2\mathbf{N}_0 \text{ and } (h-1)d \leq k. \end{cases} \qquad (2.17)$$

For the second sum in (2.15), because $d > 1$, we have $D = d(h+1) - 2 > (h-1)d$. Thus we have

$$iD > (i-1)D + (h-1)d, \qquad \forall\, i \in \mathbf{N}. \tag{2.18}$$

Therefore every integer in $\Omega(d,h)$ is covered precisely once by elements in $\Omega(d,h)$. Then let

$$\sum_{k\in\mathbf{Z}} v_k t^k \equiv \sum_{j=0}^{h-1}\sum_{i=1}^{+\infty} t^{iD+jd} = \sum_{i=1}^{+\infty}\sum_{j=0}^{h-1} t^{iD+jd}. \tag{2.19}$$

Here no any two terms in (2.19) with different indices (i,j) have the same exponent. Thus we obtain

$$v_k = \begin{cases} 1, & \text{if } k \in iD + d\mathbf{N} \text{ and } iD \le k \le iD + (h-1)d \text{ for some } i \in \mathbf{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.20}$$

Then from (2.15), (2.16) and (2.19) we obtain

$$b_q = u_{q-(d-1)} + v_{q-(d-1)}, \qquad \forall\, q \in \mathbf{Z}. \tag{2.21}$$

together with (2.17) and (2.20), it yields (2.11).

Because $D = d(h+1) - 2 > (h-1)d$, to get the sum (2.12), by (2.21) for any integers $p \ge 1$ and $0 \le m \le D-1$ we compute

$$\begin{aligned} \sum_{j=0}^{pD+m}(u_j+v_j) &= \sum_{j=0}^{h-2}(j+1)\sum_{2i=jd}^{(j+1)d-2} 1 + \left(h \sum_{(h-1)d\le 2i\le pD+m} 1\right) \\ &\quad + \sum_{i=1}^{p-1}\sum_{j=0}^{h-1} 1 + 1 + [\frac{m}{d}] - [\frac{m}{hd}] \\ &= \frac{h(h-1)}{2}\cdot\frac{d}{2} + h(\frac{pD-(h-1)d+2}{2} + [\frac{m}{2}]) \\ &\quad + (p-1)h + 1 + [\frac{m}{d}] - [\frac{m}{hd}] \\ &= h(\frac{D}{2}+1)p - \frac{h(h-1)d}{4} + 1 + h[\frac{m}{2}] + [\frac{m}{d}] - [\frac{m}{hd}], \end{aligned} \tag{2.22}$$

where on the right hand side of the first equality the first two sums come from u_js, and the third sum and the last three terms come from v_js. The number 1 there corresponds to the term t^{pD}. Note that by the fact $0 < m \le D-1$, we have $m < (h-1)d + 2d = (h+1)d$. But we may have $m \ge hd$. If this happens, the term $[m/d]$ in the right hand side of the first equality will be precisely one greater than it should be in (2.19). Thus the term $-[m/(hd)]$ is added to cancel this possible surplus 1. Then $1 + [\frac{m}{d}] - [\frac{m}{hd}]$ gives the total contribution of v_js after the power $t^{(p-1)D+(h-1)d}$.

Therefore for every integer $k \ge hd - 1$, letting

$$p = [\frac{k-(d-1)}{D}] \quad \text{and} \quad m = k - (d-1) - pD,$$

we obtain

$$m = k-(d-1)-[\frac{k-(d-1)}{D}]D = \{\frac{k-(d-1)}{D}\}D < D. \tag{2.23}$$

Then by (2.21) we obtain

$$\sum_{q=0}^{k} b_q = \sum_{q=0}^{k}(u_{q-(d-1)}+v_{q-(d-1)}) = \sum_{j=-(d-1)}^{k-(d-1)}(u_j+v_j) = \sum_{j=0}^{k-(d-1)}(u_j+v_j), \tag{2.24}$$

where (2.17), (2.20) and the fact $d \geq 2$ are used.

Thus replacing $k-(d-1) = pD+m$ with the above p and m into (2.22) and replacing $[a]$ by $a-\{a\}$ for $a \in \mathbf{R}$ below, we obtain

$$\begin{aligned}
\sum_{q=0}^{k} b_q &= \sum_{j=0}^{pD+m}(u_j+v_j) \\
&= h(\frac{D}{2}+1)[\frac{k-(d-1)}{D}] - \frac{h(h-1)d}{4} + 1 + h[\frac{k-(d-1)-[\frac{k-(d-1)}{D}]D}{2}] \\
&\quad +[\frac{k-(d-1)-[\frac{k-(d-1)}{D}]D}{d}] - [\frac{k-(d-1)-[\frac{k-(d-1)}{D}]D}{hd}] \\
&= h(\frac{D}{2}+1)\frac{k-(d-1)}{D} - \frac{h(h-1)d}{4} + 1 + h\frac{k-(d-1)-[\frac{k-(d-1)}{D}]D}{2} \\
&\quad +\frac{k-(d-1)-[\frac{k-(d-1)}{D}]D}{d} - \frac{k-(d-1)-[\frac{k-(d-1)}{D}]D}{hd} \\
&\quad -h(\frac{D}{2}+1)\{\frac{k-(d-1)}{D}\} - h\{\frac{k-(d-1)-[\frac{k-(d-1)}{D}]D}{2}\} \\
&\quad -\{\frac{k-(d-1)-[\frac{k-(d-1)}{D}]D}{d}\} + \{\frac{k-(d-1)-[\frac{k-(d-1)}{D}]D}{hd}\} \\
&= h(\frac{D}{2}+1)\frac{k-(d-1)}{D} - \frac{h(h-1)d}{4} + 1 + \frac{hD}{2}\{\frac{k-(d-1)}{D}\} \\
&\quad +\frac{D}{d}\{\frac{k-(d-1)}{D}\} - \frac{D}{hd}\{\frac{k-(d-1)}{D}\} - h(\frac{D}{2}+1)\{\frac{k-(d-1)}{D}\} \\
&\quad -h\{\frac{D}{2}\{\frac{k-(d-1)}{D}\}\} - \{\frac{D}{d}\{\frac{k-(d-1)}{D}\}\} + \{\frac{D}{hd}\{\frac{k-(d-1)}{D}\}\} \\
&= h(\frac{D}{2}+1)\frac{k-(d-1)}{D} - \frac{h(h-1)d}{4} + 1 + (\frac{D}{d} - h - \frac{D}{hd})\{\frac{k-(d-1)}{D}\} \\
&\quad +\{\frac{D}{hd}\{\frac{k-(d-1)}{D}\}\} - h\{\frac{D}{2}\{\frac{k-(d-1)}{D}\}\} - \{\frac{D}{d}\{\frac{k-(d-1)}{D}\}\}. \qquad (2.25)
\end{aligned}$$

Then from

$$\frac{D}{d} - h - \frac{D}{hd} = 1 - \frac{2}{d} - \frac{D}{hd} = -\frac{2}{d} - \frac{d-2}{hd},$$

we obtain (2.12). ∎

Remark 2.7. When d is even and $h = 1$, the first equality of (2.12) is exactly the third equality of (2.9). In fact, in this case, there holds $h = 1$ and $D = 2(d-1)$. So by (2.12)-(2.13) we have

$$\sum_{q=0}^{k} b_q|_{h=1} = \frac{h(h+1)d}{2D}(k-(d-1)) - \frac{h(h-1)d}{4} + 1 + \epsilon_{d,1}(k)$$

$$= \quad \frac{kd}{2(d-1)} - \frac{d-2}{2} - \left(\{\frac{k-(d-1)}{2(d-1)}\} + \{\frac{k-(d-1)}{2}\}\right). \tag{2.26}$$

On the other hand, by (2.9) and noting that d is even, we have

$$\begin{aligned}
&\{\frac{[\frac{k}{d-1}]+1}{2}\} + \{\frac{k+1}{2}\} + \frac{1}{2}\{\frac{k}{d-1}\} \\
&\qquad = \{\{\frac{k+(d-1)}{2(d-1)}\} - \frac{1}{2}\{\frac{k}{d-1}\}\} + \frac{1}{2}\{\frac{k}{d-1}\} + \{\frac{k-(d-1)}{2}\}.
\end{aligned} \tag{2.27}$$

Note that no matter the integer $[\frac{k}{d-1}]$ is odd or even, we have always

$$\{\frac{k+(d-1)}{2(d-1)}\} = \{\frac{1}{2}([\frac{k}{d-1}]+1) + \frac{1}{2}\{\frac{k}{d-1}\}\} \geq \frac{1}{2}\{\frac{k}{d-1}\}.$$

Thus (2.27) yields

$$\{\frac{[\frac{k}{d-1}]+1}{2}\} + \{\frac{k+1}{2}\} + \frac{1}{2}\{\frac{k}{d-1}\} = \{\frac{k+(d-1)}{2(d-1)}\} + \{\frac{k-(d-1)}{2}\}. \tag{2.28}$$

Then (2.26) and (2.28) complete the proof of the above claim.

3 Morse indices of closed geodesics

3.1 Basic normal form decompositions of symplectic matrices and precise index iteration formulae

In [Lon1] of 1999, Y. Long established the basic normal form decomposition of symplectic matrices. Based on this result he further established the precise iteration formulae of indices of symplectic paths in [Lon2] of 2000. These results form the basis of our study on the Morse indices and homological properties of closed geodesics. Here we briefly review these results:

As in [Lon3], denote by

$$N_1(\lambda, a) \quad = \quad \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}, \qquad \text{for } \lambda = \pm 1,\ a \in \mathbf{R}, \tag{3.1}$$

$$H(b) \quad = \quad \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}, \qquad \text{for } b \in \mathbf{R} \setminus \{0, \pm 1\}, \tag{3.2}$$

$$R(\theta) \quad = \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \text{for } \theta \in (0,\pi) \cup (\pi, 2\pi), \tag{3.3}$$

$$\begin{aligned}
N_2(e^{\theta\sqrt{-1}}, B) \quad &= \quad \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}, \qquad \text{for } \theta \in (0,\pi) \cup (\pi, 2\pi) \ \text{ and} \\
&\qquad\qquad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \text{ with } b_j \in \mathbf{R}, \ \text{ and } \ b_2 \neq b_3.
\end{aligned} \tag{3.4}$$

Here $N_2(e^{\theta\sqrt{-1}}, B)$ is non-trivial if $(b_2 - b_3)\sin\theta < 0$, and trivial if $(b_2 - b_3)\sin\theta > 0$ as defined in [Lon2] and Definition 1.8.11 of [Lon3]. Note that symplectic paths with end matrices in these two

cases have rather different index iteration properties as proved in [Lon2] (cf. Theorems 8.2.3 and 8.2.4 of [Lon3]). In [Lon1]-[Lon3], all the matrices listed in (3.1)-(3.4) are called **basic normal forms** of symplectic matrices.

As in [Lon3], given any two real matrices of the square block form

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}_{2i\times 2i}, \qquad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}_{2j\times 2j},$$

the $\diamond$-sum (direct sum) of M_1 and M_2 is defined by the $2(i+j)\times 2(i+j)$ matrix

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Definition 3.1. (cf. [Lon2] and [Lon3]) *For every $P \in \mathrm{Sp}(2d)$, the homotopy set $\Omega(P)$ of P in $\mathrm{Sp}(2d)$ is defined by*

$$\Omega(P) = \{N \in \mathrm{Sp}(2d) \,|\, \sigma(N) \cap \mathbf{U} = \sigma(P) \cap \mathbf{U} \equiv \Gamma \textit{ and } \nu_\omega(N) = \nu_\omega(P)\, \forall \omega \in \Gamma\},$$

where $\sigma(P)$ denotes the spectrum of P, $\nu_\omega(P) \equiv \dim_{\mathbf{C}} \ker_{\mathbf{C}}(P - \omega I)$ for all $\omega \in \mathbf{U}$. The homotopy component $\Omega^0(P)$ of P in $\mathrm{Sp}(2d)$ is defined by the path connected component of $\Omega(P)$ containing P (cf. p.38 of [Lon3]).

Note that $\Omega^0(P)$ defines an equivalent relation among symplectic matrices. Specially two matrices N and $P \in \mathrm{Sp}(2d)$ are homotopic if $N \in \Omega^0(P)$, and in this case we write $N \approx P$.

Then the following decomposition theorem is proved in [Lon1] and [Lon2]

Theorem 3.2. (cf. Theorem 7.8 of [Lon1], Lemma 2.3.5 and Theorem 1.8.10 of [Lon3]) *For every $P \in \mathrm{Sp}(2d)$, there exists a continuous path $f \in \Omega^0(P)$ such that $f(0) = P$ and*

$$\begin{aligned} f(1) \;=\; & N_1(1,1)^{\diamond p_-} \diamond I_{2p_0} \diamond N_1(1,-1)^{\diamond p_+} \\ & \diamond N_1(-1,1)^{\diamond q_-} \diamond (-I_{2q_0}) \diamond N_1(-1,-1)^{\diamond q_+} \\ & \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_k) \diamond R(\theta_{k+1}) \diamond \cdots \diamond R(\theta_r) \\ & \diamond N_2(e^{\alpha_1\sqrt{-1}}, A_1) \diamond \cdots \diamond N_2(e^{\alpha_{k_*}\sqrt{-1}}, A_{k_*}) \\ & \qquad \diamond N_2(e^{\alpha_{k_*+1}\sqrt{-1}}, A_{k_*+1}) \diamond \cdots \diamond N_2(e^{\alpha_{r_*}\sqrt{-1}}, A_{r_*}) \\ & \diamond N_2(e^{\beta_1\sqrt{-1}}, B_1) \diamond \cdots \diamond N_2(e^{\beta_{k_0}\sqrt{-1}}, B_{k_0}) \\ & \qquad \diamond N_2(e^{\beta_{k_0+1}\sqrt{-1}}, B_{k_0+1}) \diamond \cdots \diamond N_2(e^{\beta_{r_0}\sqrt{-1}}, B_{r_0}) \\ & \diamond H(2)^{\diamond h_+} \diamond H(-2)^{\diamond h_-}, \end{aligned} \tag{3.5}$$

where $\frac{\theta_j}{2\pi} \notin \mathbf{Q}$ for $1 \le j \le k$ and $\frac{\theta_j}{2\pi} \in \mathbf{Q}$ for $k+1 \le j \le r$; $N_2(e^{\alpha_j\sqrt{-1}}, A_j)$'s are nontrivial basic normal forms with $\frac{\alpha_j}{2\pi} \notin \mathbf{Q}$ for $1 \le j \le k_*$ and $\frac{\alpha_j}{2\pi} \in \mathbf{Q}$ for $k_*+1 \le j \le r_*$; $N_2(e^{\beta_j\sqrt{-1}}, B_j)$'s are

trivial basic normal forms with $\frac{\beta_j}{2\pi}\notin\mathbf{Q}$ for $1\le j\le k_0$ and $\frac{\beta_j}{2\pi}\in\mathbf{Q}$ for $k_0+1\le j\le r_0$; $p_-=p_-(P)$, $p_0=p_0(P)$, $p_+=p_+(P)$, $q_-=q_-(P)$, $q_0=q_0(P)$, $q_+=q_+(P)$, $r=r(P)$, $k=k(P)$, $r_j=r_j(P)$, $k_j=k_j(P)$ with $j=*,0$ and $h_+=h_+(P)$ are nonnegative integers, and $h_-=h_-(P)\in\{0,1\}$; θ_j, α_j, $\beta_j\in(0,\pi)\cup(\pi,2\pi)$; these integers and real numbers are uniquely determined by P and satisfy

$$p_-+p_0+p_++q_-+q_0+q_++r+2r_*+2r_0+h_-+h_+=d. \tag{3.6}$$

For $\tau>0$ and $d\in\mathbf{N}$ let

$$\mathcal{P}_\tau(2d)=\{\gamma\in C([0,\tau],\mathrm{Sp}(2d)\,|\,\gamma(0)=I\}.$$

Based on Theorem 3.2, the homotopy invariance and symplectic additivity of the indices, the following precise iteration formula was proved in [Lon2]:

Theorem 3.3. (cf. [Lon2], Theorem 8.3.1 and Corollary 8.3.2 of [Lon3]) *Let $\gamma\in\mathcal{P}_\tau(2d)$. Denote the basic normal form decomposition of $P\equiv\gamma(\tau)$ by (3.5). Then we have*

$$\begin{aligned}
i(\gamma^m) &= m(i(\gamma)+p_-+p_0-r)+2\sum_{j=1}^{r}E\left(\frac{m\theta_j}{2\pi}\right)-r\\
&\quad -p_--p_0-\frac{1+(-1)^m}{2}(q_0+q_+)\\
&\quad +2\sum_{j=k_*+1}^{r_*}\varphi\left(\frac{m\alpha_j}{2\pi}\right)-2(r_*-k_*), \qquad (3.7)\\
\nu(\gamma^m) &= \nu(\gamma)+\frac{1+(-1)^m}{2}(q_-+2q_0+q_+)+2\varsigma(m,\gamma(\tau)), \qquad (3.8)\\
\hat{i}(\gamma) &= i(\gamma)+p_-+p_0-r+\sum_{j=1}^{r}\frac{\theta_j}{\pi}, \qquad (3.9)
\end{aligned}$$

where we denote by

$$\begin{aligned}
\varsigma(m,\gamma(\tau)) &= (r-k)-\sum_{j=k+1}^{r}\varphi\left(\frac{m\theta_j}{2\pi}\right)\\
&\quad +(r_*-k_*)-\sum_{j=k_*+1}^{r_*}\varphi\left(\frac{m\alpha_j}{2\pi}\right)+(r_0-k_0)-\sum_{j=k_0+1}^{r_0}\varphi\left(\frac{m\beta_j}{2\pi}\right). \qquad (3.10)
\end{aligned}$$

By Theorems 8.1.4-8.1.7 and 8.2.1-8.2.4 on pp179-187 of [Lon3], we have specially

Proposition 3.4. *Every path $\gamma\in\mathcal{P}_\tau(2)$ with end matrix being homotopic to one of the following matrices must have odd index $i(\gamma)$,*

$$N_1(1,b_1),\quad N_1(-1,b_2),\quad R(\theta),\quad \textit{or}\quad H(-2), \tag{3.11}$$

where $b_1=0$ or 1, $b_2=0$ or ±1, and $\theta\in(0,\pi)\cup(\pi,2\pi)$. Paths $\xi\in\mathcal{P}_\tau(2)$ with end matrix being homotopic to $N_1(1,-1)$ or $H(2)$, and $\eta\in\mathcal{P}_\tau(4)$ with end matrix being homotopic to $N_2(\omega,B)$ must have even indices $i(\xi)$ and $i(\eta)$.

Remark 3.5. Note that all closed geodesics on a simply connected manifold M are orientable. Therefore, the Morse index of a closed geodesic on M equals the above Maslov-type index of a symplectic path starting from identity I and ending at $P_c \in \mathrm{Sp}(2(\dim M - 1))$ (cf. Theorem 1.1 of [Liu1] and Theorem 3 of [Wil1]). Next we will apply the precise iteration indices to study properties of Morse indices of closed geodesics.

3.2 The monotonicity of index growth

In [LoD1], closed geodesics on Finsler manifold are classified into two families, rational and irrational ones, as follows.

Definition 3.6. (cf. Definitions 3.4 and 3.6 of [LoD1]) *A matrix $P \in \mathrm{Sp}(2d)$ is* **rational***, if no basic normal form in (3.5) of P is of the form $R(\theta)$ with $\theta/\pi \in \mathbf{R}\setminus\mathbf{Q}$, and is* **irrational***, otherwise. Let $\nu(P) = \dim_{\mathbf{R}} \ker_{\mathbf{R}}(P - I)$. P is* **equally degenerate***, if $\nu(P^m) = \nu(P)$ for all $m \in \mathbf{N}$. P* **completely non-degenerate***, if $\nu(P^m) = 0$ for all $m \geq 1$.*

Let (M, F) be a d-dimensional Finsler manifold. Let c be an orientable closed geodesic on (M, F) whose linearized Poincaré map is denoted by P_c and then $P_c \in \mathrm{Sp}(2d - 2)$. The closed geodesic c is **rational**, **irrational**, **equally degenerate***, or* **completely non-degenerate***, if so is P_c. The* **analytical period** *$n(c)$ of c is defined by*

$$n(c) = \min\{k \in \mathbf{N} \,|\, \nu(c^k) = \max_{m\geq 1} \nu(c^m) \ \ and \ \ i(c^{m+k}) - i(c^m) \in 2\mathbf{Z}, \ \ \forall\, m \in \mathbf{N}\}. \tag{3.12}$$

The following is also defined in [LoD1] for any closed geodesic c on (M, F), let

$$n_0(c) = \min\{k \in \mathbf{N} \,|\, \nu(c^k) = \max_{m\geq 1} \nu(c^m)\}. \tag{3.13}$$

We have the following result.

Lemma 3.7. (cf. Lemma 3.5 of [LoD1]) *Let (M, F) be a d-dimensional Finsler manifold. Let c be an orientable closed geodesic on M whose linearized Poincaré map is denoted by P_c. There hold*

$$n(c) = n_0(c) \ \ or \ \ 2n_0(c), \tag{3.14}$$

$$n(c) = 2n_0(c) \ \ if\ and\ only\ if \ \ q_- = 0, \quad h_- = 1 \quad and \quad n_0(c) \ \ is\ odd, \tag{3.15}$$

where $q_- = q_-(P_c)$ and $h_- = h_-(P_c)$ with P_c defined in (3.5).

We need

Lemma 3.8. *Let (M, F) be a Finsler manifold and c be a prime orientable closed geodesic on M. Let $n = n(c)$ be the analytical period of c. Suppose $m \in [1, n - 1]$ satisfies*

(i) $\nu(c) < \nu(c^m) < \nu(c^n)$, and

(ii) there exists no $k \in [1, m - 1]$ satisfying $k|m$, $k|n$ and $\nu(c^k) = \nu(c^m)$.

Then $m|n$ must hold.

Remark 3.9. Lemma 3.8 is precisely Proposition 3.12 of [LoD1] when c is rational and its orbit is isolated in closed geodesic orbits in ΛM in addition. By carefully checking the proof of this Proposition 3.12, one can find that it works also for irrational prime closed geodesics, and the condition on isolatedness in closed geodesic orbits in ΛM is not necessary. Therefore we omit the details of this proof here.

Lemma 3.10. *Let (M, F) be a compact Finsler manifold. Let c be an orientable closed geodesic on M with analytical period $n = n(c)$. Then $n = n(c)$ is precisely the integer N in Lemma 2.3, i.e., there holds also*

$$K(c^{n+m}) = K(c^m), \qquad \forall\, m \geq 1, \tag{3.16}$$

Proof. In fact, by the definition (2.3) of $K(c^m)$, it suffices to prove

$$k_j^{\epsilon(c^{nl+m})}(c^{nl+m}) = k_j^{\epsilon(c^m)}(c^m), \qquad \forall\, j \in \mathbf{Z},\ l \in \mathbf{N}_0,\ 1 \leq m < n. \tag{3.17}$$

Note firstly that by the definition of $n = n(c)$, for all $l \in \mathbf{N}_0$ and $1 \leq m < n$, we have $i(c^{nl+m}) - i(c^m) \in 2\mathbf{Z}$. It then yields

$$\epsilon(c^{nl+m}) = (-1)^{i(c^{nl+m})-i(c)} = (-1)^{i(c^m)-i(c)} = \epsilon(c^m). \tag{3.18}$$

By the definition of $n = n(c)$, for these integers l and m we have also

$$\nu(c^{nl+m}) = \nu(c^m). \tag{3.19}$$

Therefore we need only to prove (3.17) for $0 \leq j \leq \nu(c^m)$.

By Lemma 3.8 we obtain some integer $p \in [1, m]$ such that both $p|m$, $p|n$, and $\nu(c^p) = \nu(c^m) = \nu(c^{nl+m})$ hold. Then $p|(nl + m)$ holds and by (iii) of lemma 2.2, we obtain

$$k_j^{\epsilon(c^{nl+m})}(c^{nl+m}) = k_j^{\epsilon(c^p)}(c^p) = k_j^{\epsilon(c^m)}(c^m), \qquad \forall\, j \in \mathbf{Z}. \tag{3.20}$$

The proof is complete. ∎

Definition 3.11. *For every matrix $P \in \mathrm{Sp}(2d)$, using its basic normal form decomposition (3.5) we define*

$$\begin{cases} \sigma(P) = r + p_+ + p_0 + q_- + q_0, \\ s(P) = r + p_- + p_0 + q_+ + q_0 + 2(r_* - k_*). \end{cases} \tag{3.21}$$

Recall that we have defined in [LoD1]:

$$p(P) = p_0(P) + p_-(P) + q_0(P) + q_+(P) + r(P) + 2r_*(P). \tag{3.22}$$

Lemma 3.12. *Let c be a closed geodesic with mean index $\hat{i}(c) > 0$ on a compact simply connected Finsler manifold (M, F) of dimension $d \geq 2$. Denote the basic normal form decomposition*

of the linearized Poincaré map P_c of c by (3.5). Denote by $n = n(c)$ the analytical period of c. Let $\sigma(c) = \sigma(P_c)$ given by Definition 3.11. Then for any even integer multiple $T > 0$ of n, we have

$$i(c^T) + \nu(c^n) = \sigma(c) \qquad \mod 2. \tag{3.23}$$

Proof. By Theorem 3.3, the definition of n, and the evenness of T, we obtain

$$\begin{aligned} i(c^T) &= T(i(c) + p_- + p_0 - r) + 2\sum_{j=1}^{r} E(\frac{T\theta_j}{2\pi}) \\ &\qquad -r - p_- - p_0 - q_0 - q_+ - 2(r_* - k_*), \qquad (3.24) \\ \nu(c^n) &= \nu(c) + q_- + 2q_0 + q_+ + 2\zeta(T, \gamma(\tau)), \qquad (3.25) \end{aligned}$$

where $\zeta(T, \gamma(\tau))$ is given by (3.10). Thus we obtain

$$\begin{aligned} i(c^T) + \nu(c^n) &= \nu(c) - r - p_- - p_0 - q_- - q_0 \qquad (\text{mod } 2) \\ &= p_- + 2p_0 + p_+ - r - p_- - p_0 - q_- - q_0 \qquad (\text{mod } 2) \\ &= \sigma(c) \qquad (\text{mod } 2). \qquad (3.26) \end{aligned}$$

This proves the lemma. ∎

When the Morse index of a prime closed geodesic on a Finsler manifold M is not too small, Morse indices of iterations of this closed geodesic satisfy the following monotonicity property.

Theorem 3.13. *Let c be a closed geodesic on a compact simply connected Finsler manifold M of dimension $d \geq 2$ satisfying*

$$i(c) + p_0 + p_- \geq q_0 + q_+ + r + 2(r_* - k_*), \tag{3.27}$$

where we denote the basic normal form decomposition of the linearized Poincaré map P_c of c by (3.5). Then there holds $i(c^{m+1}) \geq i(c^m)$ for all $m \geq 1$. In particular, the condition (3.27) holds if $i(c) \geq d - 2$.

Proof. By (3.7) in Theorem 3.3, for any $m \geq 1$, we have

$$\begin{aligned} i(c^{m+1}) - i(c^m) &= i(c) + p_- + p_0 - r + 2\sum_{j=1}^{r}\left[E\left(\frac{(m+1)\theta_j}{2\pi}\right) - E\left(\frac{m\theta_j}{2\pi}\right)\right] \\ &\qquad + \frac{(-1)^m - (-1)^{m+1}}{2}(q_0 + q_+) + 2\sum_{j=k_*+1}^{r_*}\left[\varphi\left(\frac{(m+1)\alpha_j}{2\pi}\right) - \varphi\left(\frac{m\alpha_j}{2\pi}\right)\right] \\ &\geq i(c) + p_- + p_0 - (q_0 + q_+ + r + 2(r_* - k_*)), \qquad (3.28) \end{aligned}$$

which, together with the condition (3.27), yields the desired claim.

On the other hand, by Proposition 3.4 and the homotopy invariance and symplectic additivity of the index, we have

$$i(c) = p_- + p_0 + q_- + q_0 + q_+ + r + h_- \quad (\text{mod}\, 2). \tag{3.29}$$

By (3.6) with d replaced by $d-1$, it yields $q_0+q_++r+2(r_*-k_*) \le d-1$. If $q_0+q_++r+2(r_*-k_*) = d-1$, there holds $p_- + p_0 + p_+ + q_- + 2k_* + 2r_0 + h_- + h_+ = 0$ by (3.6), which implies that $i(c) + d - 1 = 0 \pmod 2$ by (3.14). Thus by $i(c) \ge d-2$, we obtain

$$i(c) + p_- + p_0 - (q_0 + q_+ + r + 2(r_* - k_*)) \ge i(c) - (d-1) \ge 0. \tag{3.30}$$

This completes the proof of Theorem 3.13. ∎

3.3 The quasi-monotonicity of index growth

Note that the Morse indices of closed geodesics in general are not monotone if the initial Morse index is small enough. For irrational closed geodesics with enough irrational rotation terms, in this section we establish a similar property, which we call *quasi-monotonicity*, to replace the monotonicity of the indices.

For rational closed geodesics, the properties of Morse indices of their iterations have been completely understood in [LoD1]. Here, we are interested in properties of Morse indices of iterations of irrational closed geodesics. This needs properties of sequences of vectors in $\mathbf{R}^n$ uniformly distributed mod one in number theory which can be found in pages 5-6 of [GrR1]

Definition 3.14. (cf. pages 5-6 of [GrR1]) *For given $v = (v_1, \ldots, v_n) \in \mathbf{R}^n$, define v mod 1 to be the vector $\{v\} = (\{v_1\}, \ldots, \{v_n\})$. The sequence of vectors $\{u_k\}_{k\in\mathbf{N}}$ with $u_k \in \mathbf{R}^n$ is* **uniformly distributed mod one** *if for any $0 \le b_j < c_j < 1$ for $j = 1, 2, \ldots, n$, we have*

$$\lim_{n\to\infty} \frac{1}{N} \#\{k \le N \mid \{u_k\} \in \oplus[b_j, c_j)\} = \Pi_{j=1}^n (c_j - b_j).$$

Proposition 3.15 (Kronecker's result, cf. page 6 of [GrR1]) *If $1, v_1, \ldots, v_n$ are linearly independent over $\mathbf{Q}$, then the vectors $\{(kv_1, \ldots, kv_n)\}_{k\in\mathbf{N}}$ are uniformly distributed mod one on $[0,1]^n$.*

For our purpose, we need the following definition.

Definition 3.16. *Let $v = (v_1, \ldots, v_n) \in (\mathbf{R} \setminus \mathbf{Q})^n$. For a vertex $\chi \in \{0,1\}^n$ of $[0,1]^n$, we call v uniformly distributed mod one near χ, if for any given $\epsilon \in (0, 1/2)$, there exist infinitely many $m \in \mathbf{N}$ such that*

$$|\{mv\} - \chi| < \epsilon. \tag{3.31}$$

Note that when some of $1, v_1, \ldots, v_n$ are linearly dependent over $\mathbf{Q}$, the Proposition 3.15 does not hold in general. In this case, the sequence $\{\{kv\} \mid k \ge 1\}$ in general is only uniformly distributed on the intersections of some lower dimensional hyperplanes with $[0,1]^n$. For example, let $v_1 \in (0,1) \setminus \mathbf{Q}$, and $v_2 = 1 - v_1$. Then $1, v_1, v_2$ are linearly dependent over $\mathbf{Q}$, and for $v = (v_1, v_2)$ the sequence $\{\{kv\} \mid k \in \mathbf{N}\}$ is dense on the second diagonal $\{(x,y) \in [0,1]^2 \mid x + y = 1\}$ of $[0,1]^2$. Specially v is uniformly distributed mod one near the vertexes $(0,1)$ and $(1,0)$, but is not uniformly distributed mod one near the vertexes $(0,0)$ and $(1,1)$. For another extremal example: let $v_1 = v_2 \in (0,1) \setminus \mathbf{Q}$.

Then $1, v_1, v_2$ are linearly dependent over $\mathbf{Q}$, and for $v = (v_1, v_2)$ the sequence $\{\{kv\} \mid k \in \mathbf{N}\}$ is dense on the diagonal $\{(x, y) \in [0,1]^2 \mid x = y\}$ of $[0,1]^2$. Specially v is uniformly distributed mod one near the vertexes $(0,0)$ and $(1,1)$, but is not uniformly distributed mod one near the vertexes $(1,0)$ and $(0,1)$.

We need the following Theorem 11.1.2 of [Lon3] (originally proved as Theorem 4.2 of [LoZ1]) to continue our study.

Proposition 3.17. (Y. Long and C. Zhu [LoZ1]) *Fix $v = (v_1, \ldots, v_n) \in \mathbf{R}^n$. Let H be the closure of the subset $\{\{mv\} \mid m \in \mathbf{N}\}$ in $\mathbf{T}^n$ and $V = T_0\pi^{-1}H$ be the tangent space of $\pi^{-1}H$ at the origin in $\mathbf{R}^n$, where $\pi : \mathbf{R}^n \to \mathbf{T}^n$ is the projection map. Define*

$$A(v) = V \setminus \cup_{v_k \in \mathbf{R}\setminus\mathbf{Q}} \{x = (x_1, \ldots, x_n) \in V \mid x_k = 0\}.$$

Define $\psi(x) = 0$ when $x \geq 0$ and $\psi(x) = 1$ when $x < 0$. Then for any $a = (a_1, \ldots, a_n) \in A(v)$, the vector

$$\chi = (\psi(a_1), \ldots, \psi(a_n))$$

makes

$$|\{Nv\} - \chi| < \epsilon.$$

holds for infinitely many $N \in \mathbf{N}$.

Moreover, this set $A(v)$ possesses the following properties.

(a) $A(v) \neq \emptyset$.

(b) When $v \in \mathbf{Q}^n$, there holds $V = A(v) = \{0\}$.

(c) When $v \in \mathbf{R}^n \setminus \mathbf{Q}^n$, there hold $\dim V \geq 1$, $0 \notin A(v) \subset V$, $A(v) = -A(v)$, and that $A(v)$ is open in V.

(d) When $\dim V = 1$, there holds $A(v) = V \setminus \{0\}$.

(e) When $\dim V \geq 2$, $A(v)$ is obtained from V by deleting all the coordinate hyperplanes with dimension strictly smaller than $\dim V$ from V, and specially $\dim A(v) = \dim V$.

Denote by $\hat{1} = (1, \ldots, 1) \in \mathbf{R}^n$. Define the opposite vertex $\hat{\chi}$ of a vertex χ in $[0,1]^n$ by

$$\hat{\chi} = \hat{1} - \chi. \tag{3.32}$$

The following lemma is a generalization of the above example and will be useful later.

Lemma 3.18. *Let $v = (v_1, \ldots, v_n) \in (\mathbf{R} \setminus \mathbf{Q})^n$ be uniformly distributed mod one near a vertex $\chi \in \{0,1\}^n$ of $[0,1]^n$. Then v is also uniformly distributed mod one near the opposite vertex $\hat{\chi}$ of χ.*

Proof. For the given $v \in (\mathbf{R} \setminus \mathbf{Q})^n$, we apply Proposition 3.17 to prove the lemma. Using notations in Proposition 3.17, we obtain $A(v) \neq \emptyset$ by the conclusion (a) and the fact $v \in (\mathbf{R} \setminus \mathbf{Q})^n$.

Now using the function $\psi : \mathbf{R} \to \{0,1\}$ in Proposition 3.17, we further define a map $\hat{\psi}$ from $A(v)$ to vertexes of $[0,1]^n$ by

$$\hat{\psi}(a) = (\psi(a_1), \ldots, \psi(a_n)), \qquad \forall\, a = (a_1, \ldots, a_n) \in A(v).$$

Then we have the following two claims:

Claim (i) *If v is uniformly distributed mod one near a vertex χ of $[0,1]^n$, then there exists an $a \in A(v)$ such that $\chi = \hat{\psi}(a)$.*

In fact, let H_0 be the closure of the set $\{\{mv\} \,|\, m \in \mathbf{N}\}$ in $[0,1]^n$. It is well known that the closed set H_0 consists of only finitely many connected components which are intersections of parallel equal dimensional subspaces with $[0,1]^n$ (cf. descriptions in Sections 23.4 on page 508 and 23.10 on page 522 of [HaW1]) and determined by the integral linearly dependent relations satisfied by the irrational numbers $\{v_1, \ldots, v_n\}$. Then we have $H = \pi(H_0)$ and V in Proposition 3.17 can be identified as the linear subspace of $\mathbf{R}^n$ passing through 0, parallel to H_0, and satisfying $\dim V = \dim H_0$. Because the closed set H_0 consists of only finitely many connected components, we can choose an $\epsilon \in (0, 1/4)$ sufficiently small such that the ball $B_\epsilon(\chi)$ with radius ϵ centered at the point χ in $\mathbf{R}^n$ has non-empty intersection with only one connected component H_1 of H_0. Because v is uniformly distributed mod one near the vertex χ, we can choose a sufficiently large $m \in \mathbf{N}$ such that $\{mv\} \in B_\epsilon(\chi) \cap H_1$. Denote the point $\{mv\}$ by $b = (b_1, \ldots, b_n)$. Then by the choice of ϵ, we have either $b_i \in (0, 1/4)$ or $b_i \in (3/4, 1)$ for each $i = 1, \ldots, n$. Here the fact $v \in (\mathbf{R} \setminus \mathbf{Q})^n$ is used. We define a new point $a = (a_1, \ldots, a_n)$ by

$$a_i = \begin{cases} b_i, & \text{if } \ b_i \in (0, 1/4), \\ b_i - 1, & \text{if } \ b_i \in (3/4, 1), \end{cases}$$

for all $i = 1, \ldots, n$. Denote the segment connecting χ to b by l_1, and the straight line passing through 0 and parallel to l_1 by l_2 (cf. The definitions of a and b in Figure 3.1).

Then by the definitions of V and $A(v)$, we have

$$a \;\in\; l_2 \setminus \{0\} \;\subset\; A(v) \;\subset\; V.$$

Specially we have $\hat{\psi}(a) = \chi$ and proves the Claim (i).

Claim (ii) *If a vertex χ of $[0,1]^n$ is in the image of $\hat{\psi}$, so is its opposite vertex $\hat{\chi}$ in $[0,1]^n$.*

In fact, let $a = (a_1, \ldots, a_n) \in A(v)$ and $\chi = \hat{\psi}(a)$. Then $-a \in A(v)$ by (c) of Proposition 3.17 and $a_i \neq 0$ for all $1 \le i \le n$ by (d) and (e) of Proposition 3.17. Therefore $\hat{\chi} = \hat{\psi}(-a)$ holds, which completes the proof of Claim (ii).

Now Lemma 3.18 follows from these two claims. ∎

Corollary 3.19. *Let $v = (v_1, \ldots, v_n) \in (\mathbf{R} \setminus \mathbf{Q})^n$. Then there exists an integer r satisfying $[(n+1)/2] \le r \le n$ and a subset P of $\{1, \ldots, n\}$ containing r integers, such that for any $\epsilon \in (0, 1/4)$*

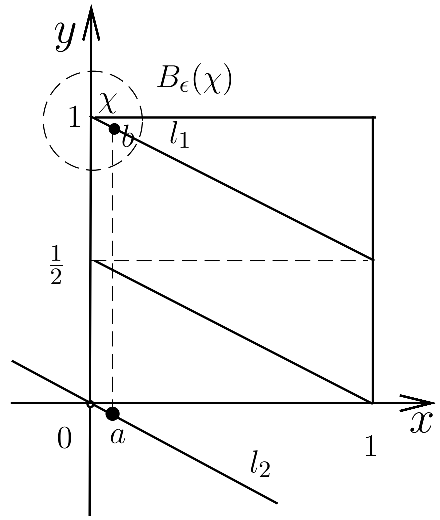

Figure 3.1: The points a and b in the 2 dimensional case.

there exist infinitely many integers T_1 and $T_2 \in \mathbf{N}$ satisfying respectively

$$\begin{cases} \{T_1 v_i\} > 1-\epsilon, & \text{for } i \in P, \\ \{T_1 v_j\} < \epsilon, & \text{for } j \in \{1,\dots,n\} \setminus P, \end{cases} \tag{3.33}$$

and

$$\begin{cases} \{T_2 v_i\} < \epsilon, & \text{for } i \in P, \\ \{T_2 v_j\} > 1-\epsilon, & \text{for } j \in \{1,\dots,n\} \setminus P. \end{cases} \tag{3.34}$$

Proof. By the proof of Theorem 4.1 of [LoZ1] (cf. pp.233-234 of [Lon3]), there exists a vertex χ of $[0,1]^n$ such that v is uniformly distributed mod one near χ. By Lemma 3.18, v is also uniformly distributed mod one near the opposite vertex $\hat{\chi}$ of χ in $[0,1]^n$. Let $P(\xi) = \{j \in \{1,\dots,n\} \,|\, \xi_j = 1\}$ for any $\xi \in \{0,1\}^n$. Let P be the one of $P(\chi)$ and $P(\hat{\chi})$ which contains not fewer integers. Let $r = {}^\# P$. Then the conclusion of Corollary 3.19 follows. ∎

To estimate Morse indices of closed geodesics, we need first

Definition 3.20. *For a prime orientable closed geodesic c with mean index $\hat{i}(c) > 0$ on a Finsler manifold (M,F) of dimension $d \geq 2$. Denote the basic normal form decomposition of the linearized Poincaré map P_c of c by (3.5). Using $\lambda = i(c) + p_- + p_0 - r$ and $\rho(m) = \sum_{j=1}^r [\frac{m\theta_j}{2\pi}]$ for any integer $A \in [0,k]$ with k given in (3.5), we define*

$$\chi_c(m) = m\lambda + 2\rho(m), \qquad \forall\, m \in \mathbf{N}, \tag{3.35}$$

$$m_1(c) = \min\{\hat{m} \in \mathbf{N} \mid \chi_c(m) \geq i(c) + 4\dim M + 2k, \text{ when } m \geq \hat{m}\}, \tag{3.36}$$

$$\alpha_A(c) = \min\left\{ \left\{\frac{m\theta_j}{2\pi}\right\} \,\middle|\, 1 \leq j \leq A,\, 1 \leq m \leq m_1(c) \right\}, \tag{3.37}$$

$$\beta_A(c) = \min\left\{ \left\{\frac{m\theta_j}{2\pi}\right\} \,\middle|\, A+1 \leq j \leq k,\, 1 \leq m \leq m_1(c) \right\}. \tag{3.38}$$

Here we have $\alpha_A(c)$ and $\beta_A(c) \in (0,1)$ whenever they are defined. Note that from $\hat{i}(c) = \lambda + \sum_{j=1}^{r} \frac{\theta_j}{\pi} > 0$, we obtain

$$
\begin{aligned}
\chi_c(m) &= m\lambda + 2\sum_{j=1}^{r}[\frac{m\theta_j}{2\pi}] \\
&= m(\lambda + \sum_{j=1}^{r}\frac{\theta_j}{\pi}) - 2\sum_{j=1}^{r}\{\frac{m\theta_j}{2\pi}\} \\
&= m\hat{i}(c) - 2\sum_{j=1}^{r}\{\frac{m\theta_j}{2\pi}\} \\
&\geq m\hat{i}(c) - 2r. \qquad (3.39)
\end{aligned}
$$

Thus the positive integer $m_1(c)$ in (3.36) and then $\alpha_A(c)$ and $\beta_A(c)$ are well defined and depend only on c , because $\hat{i}(c) > 0$.

The following is our main estimate in this section.

Theorem 3.21. (**Quasi-monotonicity of index growth for irrational closed geodesics**) *Let c be a closed geodesic with mean index $\hat{i}(c) > 0$ on a compact simply connected Finsler manifold (M, F) of dimension $d \geq 2$. Denote the basic normal form decomposition of the linearized Poincaré map P_c of c by (3.5). Then there exist an integer A with $[(k+1)/2] \leq A \leq k$ and a subset P of integers $\{1, \ldots, k\}$ with A integers such that for any $\epsilon \in (0, 1/4)$ there exists an sufficiently large integer $T \in n\mathbf{N}$ satisfying*

$$
\begin{aligned}
\left\{\frac{T\theta_j}{2\pi}\right\} &> 1-\epsilon, \qquad \text{for } j \in P, \qquad (3.40) \\
\left\{\frac{T\theta_j}{2\pi}\right\} &< \epsilon, \qquad \text{for } j \in \{1, \ldots, k\} \setminus P. \qquad (3.41)
\end{aligned}
$$

Consequently we have

$$
\begin{aligned}
i(c^m) - i(c^T) &\geq K_1 \equiv \lambda + (q_0 + q_+) + 2(r-k) + 2(r_* - k_*) + 2A, \quad \forall\, m \geq T+1, \quad (3.42) \\
i(c^T) - i(c^m) &\geq K_2 \equiv \lambda - (q_0 + q_+) + 2k - 2(r_* - k_*) - 2A, \quad \forall\, 1 \leq m \leq T-1, \quad (3.43)
\end{aligned}
$$

where $\lambda = i(c) + p_- + p_0 - r$, the integers p_-, p_0, q_0, q_+, r, k, r_* and k_* are defined in (3.5).

Remark 3.22. Similar to our discussion after Definition 3.16, in Theorem 3.21 we proved the existence of the integer A located inside the interval $[[(k+1)/2], k]$. But we do not know in general which precise value it may take without further knowledge on the θ_j/πs. Specially if $k \geq 2$ and these irrational numbers are linearly dependent over $\mathbf{Q}$, then A can not take every integer value between 1 and k.

Proof of Theorem 3.21. We carry out the proof in several steps.

Step 1. Note first that iteration formulae of Morse indices of symplectic paths ending at $N_1(-1,-1)$ or $N_1(-1,0)$ and those ending at $R(\pi)$ are precisely the same, although their nullity

may be different by 1 (cf. Sections 8.1 and 8.2 of [Lon3]). Because our current theorem concerns only Morse indices of iterations of a closed geodesic, so for simplicity of the description we shall replace all terms of $N_1(-1,-1)$ and $N_1(-1,0)$ by $R(\pi)$s in (3.5) and thus replace r by the value of $r+q_0+q_+$ and set $\frac{1+(-1)^m}{2}(q_0+q_+)=0$ in (3.7).

Step 2. *Reduction to estimates on $\chi_c(m)$.*

Therefore, by (3.5), Theorem 3.3 and Step 1, for any $m\geq 1$, the iteration formulae of Morse indices of this closed geodesic c is given by

$$\begin{aligned} i(c^m) &= m(i(c)+p_-+p_0-r)+2\sum_{j=1}^{r}E\left(\frac{m\theta_j}{2\pi}\right) \\ &\quad -r-p_--p_0-2(r_*-k_*)+2\sum_{j=k_*+1}^{r_*}\varphi\left(\frac{m\alpha_j}{2\pi}\right). \end{aligned} \tag{3.44}$$

Therefore for any $T\in\mathbf{N}$, we obtain

$$\begin{aligned} i(c^{m+T}) &- i(c^T) \\ &= m\lambda+2\sum_{j=1}^{r}\left[E\left(\frac{(m+T)\theta_j}{2\pi}\right)-E\left(\frac{T\theta_j}{2\pi}\right)\right]+2\sum_{j=k_*+1}^{r_*}\left[\varphi\left(\frac{(m+T)\alpha_j}{2\pi}\right)-\varphi\left(\frac{T\alpha_j}{2\pi}\right)\right] \\ &= m\lambda+2\left(\sum_{j=1}^{r}\left[\frac{m\theta_j}{2\pi}\right]+\sum_{j=1}^{r}\left[E\left(\left\{\frac{T\theta_j}{2\pi}\right\}+\left\{\frac{m\theta_j}{2\pi}\right\}\right)-E\left(\left\{\frac{T\theta_j}{2\pi}\right\}\right)\right]\right) \\ &\quad +2\sum_{j=k_*+1}^{r_*}\left[\varphi\left(\left\{\frac{T\alpha_j}{2\pi}\right\}+\left\{\frac{m\alpha_j}{2\pi}\right\}\right)-\varphi\left(\left\{\frac{T\alpha_j}{2\pi}\right\}\right)\right],\quad \forall\, m\geq 1, \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} i(c^T) &- i(c^{T-m}) \\ &= m\lambda+2\sum_{j=1}^{r}\left[E\left(\frac{T\theta_j}{2\pi}\right)-E\left(\frac{(T-m)\theta_j}{2\pi}\right)\right]+2\sum_{j=k_*+1}^{r_*}\left[\varphi\left(\frac{T\alpha_j}{2\pi}\right)-\varphi\left(\frac{(T-m)\alpha_j}{2\pi}\right)\right] \\ &= m\lambda+2\left(\sum_{j=1}^{r}\left[\frac{m\theta_j}{2\pi}\right]+\sum_{j=1}^{r}\left[E\left(\left\{\frac{T\theta_j}{2\pi}\right\}\right)-E\left(\left\{\frac{T\theta_j}{2\pi}\right\}-\left\{\frac{m\theta_j}{2\pi}\right\}\right)\right]\right) \\ &\quad +2\sum_{j=k_*+1}^{r_*}\left[\varphi\left(\left\{\frac{T\alpha_j}{2\pi}\right\}\right)-\varphi\left(\left\{\frac{T\alpha_j}{2\pi}\right\}-\left\{\frac{m\alpha_j}{2\pi}\right\}\right)\right],\quad \forall\, 1\leq m\leq T-1. \end{aligned} \tag{3.46}$$

For $j=1,\ldots,r$, $i=1,\ldots,k_*$ and $m\geq 1$, let

$$\mathcal{E}_j^{\pm}(T,m) = E\left(\left\{\frac{T\theta_j}{2\pi}\right\}\pm\left\{\frac{m\theta_j}{2\pi}\right\}\right), \tag{3.47}$$

$$\mathcal{E}_j(T) = E\left(\left\{\frac{T\theta_j}{2\pi}\right\}\right), \tag{3.48}$$

$$\varphi_i^{\pm}(T,m) = \varphi\left(\left\{\frac{T\alpha_i}{2\pi}\right\}\pm\left\{\frac{m\alpha_i}{2\pi}\right\}\right), \tag{3.49}$$

$$\varphi_i(T) = \varphi\left(\left\{\frac{T\alpha_i}{2\pi}\right\}\right). \tag{3.50}$$

Using these notations, from (3.45) and (3.46) we obtain

$$\begin{aligned} i(c^{m+T}) - i(c^T) &= \chi_c(m) + 2\sum_{j=1}^{r}(\mathcal{E}_j^+(T,m) - \mathcal{E}_j(T)) + 2\sum_{j=k_*+1}^{r_*}(\varphi_j^+(T,m) - \varphi_j(T)), \\ &\quad \forall\, m \ge 1, \qquad (3.51) \\ i(c^T) - i(c^{T-m}) &= \chi_c(m) + 2\sum_{j=1}^{r}(\mathcal{E}_j(T) - \mathcal{E}_j^-(T,m)) + 2\sum_{j=k_*+1}^{r_*}(\varphi_j(T) - \varphi_j^-(T,m)), \\ &\quad \forall\, 1 \le m \le T-1. \qquad (3.52) \end{aligned}$$

Therefore (3.42) and (3.43) are equivalent to the following estimates:

$$\begin{aligned} \chi_c(m) &+ 2\sum_{j=1}^{r}(\mathcal{E}_j^+(T,m) - \mathcal{E}_j(T)) + 2\sum_{j=k_*+1}^{r_*}(\varphi_j^+(T,m) - \varphi_j(T)) \ge K_1, \\ &\quad \forall\, m \ge 1, \qquad (3.53) \\ \chi_c(m) &+ 2\sum_{j=1}^{r}(\mathcal{E}_j(T) - \mathcal{E}_j^-(T,m)) + 2\sum_{j=k_*+1}^{r_*}(\varphi_j(T) - \varphi_j^-(T,m)) \ge K_2, \\ &\quad \forall\, 1 \le m \le T-1. \qquad (3.54) \end{aligned}$$

By the choice of $T \in n\mathbf{N}$, where $n = n(c)$ is the analytical period of c, we have

$$\begin{aligned} \mathcal{E}_j(T) &= 1, \quad 1 \le \mathcal{E}_j^+(T,m) \le 2, \quad 0 \le \mathcal{E}_j^-(T,m) \le 1, \qquad \forall\, 1 \le j \le k, \qquad (3.55) \\ \mathcal{E}_j(T) &= 0, \quad 0 \le \mathcal{E}_j^+(T,m) \le 1, \quad \mathcal{E}_j^-(T,m) = 0, \qquad \forall\, k+1 \le j \le r, \;\; m \ge 1, \qquad (3.56) \\ \varphi_j(T) &= 0, \quad \varphi_j^{\pm}(T,m) = \varphi\left(\frac{m\alpha_j}{2\pi}\right) \in \{0,1\}, \qquad \forall\, k_*+1 \le j \le r_*, \;\; m \ge 1. \qquad (3.57) \end{aligned}$$

Therefore (3.53) and (3.54) are equivalent to the following estimates:

$$\begin{aligned} \chi_c(m) &+ 2\sum_{j=1}^{r}\mathcal{E}_j^+(T,m) - 2k + 2\sum_{j=k_*+1}^{r_*}\varphi\left(\frac{m\alpha_j}{2\pi}\right) \ge K_1, \\ &\quad \forall\, m \ge 1, \qquad (3.58) \\ \chi_c(m) &+ 2k - 2\sum_{j=1}^{k}\mathcal{E}_j^-(T,m) - 2\sum_{j=k_*+1}^{r_*}\varphi\left(\frac{m\alpha_j}{2\pi}\right) \ge K_2, \\ &\quad \forall\, 1 \le m \le T-1. \qquad (3.59) \end{aligned}$$

Step 3. *Definition of the set S.*

Note that by the Definition 3.20 of $m_1(c)$ and the definitions of K_1 and K_2 in (3.42) and (3.43), we have

$$\chi_c(m) \ge i(c) + 4\dim M + 2k \ge \max\{K_1, K_2 + 2(r_* - k_*)\}, \qquad \forall\, m \ge m_1(c). \qquad (3.60)$$

Thus together with (3.55)-(3.57), the estimates (3.58) and (3.59) hold for all $m \ge m_1(c)$. Therefore to continue the proof, it suffices to find T so that (3.58) and (3.59) hold for those $m \in \mathbf{N}$ satisfying

$$1 \le m \le m_1(c) \quad \text{and} \quad \chi_c(m) < \max\{K_1, K_2 + 2(r_* - k_*)\}. \qquad (3.61)$$

According to (3.61), let

$$\hat{K} \equiv \max\{K_1, K_2 + 2(r_* - k_*)\} - \lambda. \tag{3.62}$$

Then

$$\hat{K} = \max\{2(r - k + r_* - k_*) + 2A,\ 2(k - (r_* - k_*)) - 2A\} > 0,$$

where we have used the fact $q_0 + q_+ = 0$ from Step 1 and the definitions of K_1 and K_2 in (3.42) and (3.43).

For every integer $\mu \in [0, \hat{K}]$, let

$$\begin{aligned} \mathcal{S}_\mu &= \{m \in \mathbf{N} \mid \chi_c(m) = \lambda + \mu,\ 1 \le m \le m_1(c)\}, \qquad (3.63)\\ \mathcal{S} &\equiv \bigcup_{0\le\mu\le\hat{K}} \mathcal{S}_\mu. \qquad (3.64)\end{aligned}$$

For $m \in [1, m_1(c)] \setminus S$, we have

$$\chi_c(m) > \lambda + \hat{K} \ge \max\{K_1, K_2 + 2(r_* - k_*)\}, \qquad \forall\, 1 \le m \le m_1(c). \tag{3.65}$$

Therefore together with (3.55)-(3.57), the estimates (3.58) and (3.59) hold for all $m \in [1, m_1(c)] \setminus S$.

Now it suffices to prove the estimates (3.58) and (3.59) for every $m \in S$.

Step 4. *Determinations of A and T.*

Now by Corollary 3.19, there exists an integer A with $[(k+1)/2] \le A \le k$ and a subset P of $\{1, \ldots, k\}$ with precisely A integers satisfying the conditions (3.40) and (3.41). Here specially the existence of the integer T follows from the mod one uniformly distribution property (cf. [GrR1]) used in Corollary 3.19. For notational simplicity, by reordering θ_js, without loss of generality we assume $P = \{1, \ldots, A\}$ in the following.

Thus for $\alpha_c(A)$ and $\beta_c(A) > 0$ defined in (3.37) and (3.38), we can find $T \in n\mathbf{N}$ such that

$$\begin{aligned} \left\{\frac{T\theta_j}{2\pi}\right\} &> 1 - \alpha_c(A), \qquad 1 \le j \le A, \qquad (3.66)\\ \left\{\frac{T\theta_j}{2\pi}\right\} &< \beta_c(A), \qquad A + 1 \le j \le k. \qquad (3.67)\end{aligned}$$

Thus by (3.66) and the definition (3.47) of $\mathcal{E}_j^+(T, m)$ for $1 \le m \le m_1(c)$, we obtain

$$1 < \{\frac{T\theta_j}{2\pi}\} + \alpha_c(A) < 2.$$

Then for $1 \le j \le A$ we have

$$\begin{aligned} \mathcal{E}_j^+(T, m) &= E\left(\left\{\frac{T\theta_j}{2\pi}\right\} + \left\{\frac{m\theta_j}{2\pi}\right\}\right)\\ &\ge E\left(\left\{\frac{T\theta_j}{2\pi}\right\} + \alpha_c(A)\right)\\ &= 2. \qquad (3.68)\end{aligned}$$

Therefore by (3.55) for such a j we obtain $\mathcal{E}_j^+(T,m)=2$, and then

$$\sum_{j=1}^{r}\mathcal{E}_j^+(T,m)\geq\sum_{j=1}^{k}\mathcal{E}_j^+(T,m)\geq 2A+(k-A)=k+A. \tag{3.69}$$

Similarly by (3.67) and the definition (3.47) of $\mathcal{E}_j^-(T,m)$ for $1\leq m\leq m_1(c)$, we obtain

$$-1<\left\{\frac{T\theta_j}{2\pi}\right\}-\beta_c(A)<0.$$

Then for $A+1\leq j\leq k$ we have

$$\begin{aligned}\mathcal{E}_j^-(T,m) &= E\left(\left\{\frac{T\theta_j}{2\pi}\right\}-\left\{\frac{m\theta_j}{2\pi}\right\}\right)\\ &\leq E\left(\left\{\frac{T\theta_j}{2\pi}\right\}-\beta_c(A)\right)\\ &= 0.\end{aligned} \tag{3.70}$$

Therefore by (3.55) for such a j we obtain $\mathcal{E}_j^-(T,m)=0$, and then

$$\sum_{j=1}^{k}\mathcal{E}_j^-(T,m)\leq\sum_{j=1}^{A}\mathcal{E}_j^-(T,m)\leq A. \tag{3.71}$$

On the other hand, note that because $\frac{\theta_j}{\pi}\notin\mathbf{Q}$ holds for $j=1,\ldots,k$, we obtain

$$\sum_{j=1}^{k}E\left(\left\{\frac{m\theta_j}{2\pi}\right\}\right)=k,\qquad\forall\, m\geq 1. \tag{3.72}$$

Note that

$$\begin{aligned}i(c^m) &= m\lambda+2\sum_{j=1}^{r}E\left(\frac{m\theta_j}{2\pi}\right)-r-p_--p_0-2(r_*-k_*)+2\sum_{j=k_*+1}^{r_*}\varphi\left(\frac{m\alpha_j}{2\pi}\right)\\ &= \chi_c(m)+2\sum_{j=1}^{r}E\left(\{\frac{m\theta_j}{2\pi}\}\right)-r-p_--p_0-2(r_*-k_*)+2\sum_{j=k_*+1}^{r_*}\varphi\left(\frac{m\alpha_j}{2\pi}\right).\end{aligned} \tag{3.73}$$

Thus by (3.72) for every $m\in\mathbf{N}$ we obtain

$$\begin{aligned}\chi_c(m) &= -2\sum_{j=1}^{r}E\left(\left\{\frac{m\theta_j}{2\pi}\right\}\right)+(r+p_-+p_0)+2(r_*-k_*)\\ &\quad -2\sum_{j=k_*+1}^{r_*}\varphi\left(\frac{m\alpha_j}{2\pi}\right)+i(c^m)\\ &= \lambda+(i(c^m)-i(c))+2(r-k-\sum_{j=k+1}^{r}E\left(\left\{\frac{m\theta_j}{2\pi}\right\}\right))\\ &\quad +2(r_*-k_*-\sum_{j=k_*+1}^{r_*}\varphi\left(\frac{m\alpha_j}{2\pi}\right)).\end{aligned} \tag{3.74}$$

Step 5. *Estimates (3.58) and (3.59) for $m \in S$.*

By the definition of $m \in S_\mu$ with $0 \le \mu \le \hat{K}$, we have $\chi_c(m) = \lambda + \mu$. Thus by (3.74) for such an $m \in S_\mu$, (3.58) and (3.59) are equivalent to the following estimates:

$$\begin{aligned}
\sum_{j=1}^{r} \mathcal{E}_j^+(T,m) &\ge k - \frac{\lambda}{2} - \frac{\mu}{2} - \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) + \frac{K_1}{2}, \forall 1 \le m \le m_1(c), && (3.75)\\
\sum_{j=1}^{k} \mathcal{E}_j^-(T,m) &\le k + \frac{\lambda}{2} + \frac{\mu}{2} - \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) - \frac{K_2}{2}, \forall 1 \le m \le \min\{T-1, m_1(c)\}. && (3.76)
\end{aligned}$$

We continue the study in three sub-steps according to the value of μ for (3.75) and (3.76).

Sub-step 1. *Study on (3.75) for $m \in S_\mu$ with $1 \le \mu \le 2(r - k + r_* - k_*)$.*

We start from the following

Claim 1: *For any $m \in \mathcal{S}_\mu$ with $0 \le \mu \le 2(r - k + r_* - k_*)$, the set*

$$\left\{ \left\{\frac{m\theta_j}{2\pi}\right\}, \left\{\frac{m\alpha_l}{2\pi}\right\} \,\middle|\, k+1 \le j \le r,\ k_*+1 \le l \le r_* \right\} \tag{3.77}$$

contains at least $(r - k + r_* - k_*) - [\mu/2]$ non-zero elements.

In fact, if the claim does not hold, then the number of zero elements in $\mathcal{S}_\mu$ is at least $[\mu/2] + 1$.

By the Bott formula (cf. [Bot1] and Section 12.1 of [Lon3]), there always holds $i(c^m) - i(c) \ge 0$ for all $m \in \mathbf{N}$. Thus by the definition of $\mathcal{S}_\mu$, (3.74) and the above assumption we obtain

$$\begin{aligned}
\lambda + \mu &= \chi_c(m)\\
&= \lambda + 2\left(r - k - \sum_{j=k+1}^{r} E\left(\left\{\frac{m\theta_j}{2\pi}\right\}\right)\right) + 2\left(r_* - k_* - \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right)\right) + i(c^m) - i(c)\\
&\ge \lambda + 2([\mu/2] + 1) + i(c^m) - i(c)\\
&\ge \lambda + \mu + 1 + i(c^m) - i(c). && (3.78)
\end{aligned}$$

This contradiction proves Claim 1.

Note that in this Sub-step 1, $\{\frac{T\theta_j}{2\pi}\} = 0$ for $k+1 \le j \le r$ by the choice of $T \in n\mathbf{N}$. Thus by Claim 1 we have

$$\begin{aligned}
\sum_{j=k+1}^{r} \mathcal{E}_j^+(T,m) &+ \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right)\\
&= \sum_{j=k+1}^{r} E(\{\frac{m\theta_j}{2\pi}\}) + \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right)\\
&\ge (r - k + r_* - k_*) - [\frac{\mu}{2}]. && (3.79)
\end{aligned}$$

Therefore by (3.69) and (3.79), we obtain

$$\sum_{j=1}^{r} \mathcal{E}_j^+(T,m) = \sum_{j=1}^{k} \mathcal{E}_j^+(T,m) + \left[\sum_{j=k+1}^{r} \mathcal{E}_j^+(T,m) + \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right)\right] - \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right)$$

$$
\begin{aligned}
&\geq k + A + (r - k + r_* - k_*) - [\frac{\mu}{2}] - \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) \\
&\geq k - \frac{\lambda}{2} - \frac{\mu}{2} - \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) + \frac{K_1}{2} \\
&\qquad + \left(\frac{\lambda}{2} + A + (r - k + r_* - k_*) - \frac{K_1}{2}\right). \qquad (3.80)
\end{aligned}
$$

Therefore to get (3.75), we need to require the last line in the right hand side of (3.80) to be non-negative. Thus the largest value which K_1 can take to guarantee (3.75) is:

$$
K_1 = \lambda + 2(r - k + r_* - k_* + A). \qquad (3.81)
$$

Sub-step 2. *Study on (3.75) for $m \in S_\mu$ with $2(r - k + r_* - k_*) < \mu \leq \hat{K}$.*

In this case, by (3.69) we have

$$
\begin{aligned}
\sum_{j=1}^{r} \mathcal{E}_j^+(T, m) &= \sum_{j=1}^{k} \mathcal{E}_j^+(T, m) + \left[\sum_{j=k+1}^{r} \mathcal{E}_j^+(T, m) + \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right)\right] - \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) \\
&\geq \sum_{j=1}^{k} \mathcal{E}_j^+(T, m) - \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) \\
&\geq k + A - \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) \\
&= k - \frac{\lambda}{2} - \frac{\mu}{2} - \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) + \frac{K_1}{2} + \left(\frac{\lambda}{2} + A + \frac{\mu}{2} - \frac{K_1}{2}\right). \qquad (3.82)
\end{aligned}
$$

Therefore the choice of K_1 by (3.81) yields also the largest value which K_1 can take to guarantee $\frac{\lambda}{2} + A + \frac{\mu}{2} - \frac{K_1}{2} \geq 0$ in the right hand side of (3.82), and then (3.75).

Sub-step 3. *(3.76) for $m \in [1, \min\{T - 1, m_1(c)\}] \cap \mathcal{S}_\mu$ with $0 \leq \mu \leq \hat{K}$.*

In this case, for any $m \in [1, \min\{T - 1, m_1(c)\}] \cap \mathcal{S}_\mu$ with $0 \leq \mu \leq \hat{K}$, by (3.55) and (3.71) we have

$$
\begin{aligned}
\sum_{j=1}^{k} \mathcal{E}_j^-(T, m) &\leq A \\
&= k + \frac{\lambda}{2} + \frac{\mu}{2} - \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) - \frac{K_2}{2} \\
&\qquad + A - k - \frac{\lambda}{2} - \frac{\mu}{2} + \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) + \frac{K_2}{2} \\
&\leq k + \frac{\lambda}{2} + \frac{\mu}{2} - \sum_{j=k_*+1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) - \frac{K_2}{2} \\
&\qquad + \left(A - k - \frac{\lambda}{2} - \frac{\mu}{2} + (r_* - k_*) + \frac{K_2}{2}\right). \qquad (3.83)
\end{aligned}
$$

Therefore to make

$$A - k - \frac{\lambda}{2} - \frac{\mu}{2} + (r_* - k_*) + \frac{K_2}{2} \le 0,$$

the largest value which K_2 can take is

$$K_2 = \lambda + 2(k - A) - 2(r_* - k_*). \tag{3.84}$$

From (3.83) we obtain that the choice (3.84) of K_2 makes (3.76) holds.

Now (3.81) and (3.84) make (3.75) and (3.76) hold, and complete the proof.

Step 6. As the final step, we come back to the discussion in the Step 1 of this proof, i.e., we consider the quantity $q_0 + q_+$ in (3.7) and the constants K_1 and K_2. Then replacing r by $r + q_0 + q_+$ in (3.81) and (3.84) we obtain

$$\begin{aligned} K_1 &= (\lambda - (q_0 + q_+)) + 2(r + (q_0 + q_+) - k + r_* - k_* + A) \\ &= \lambda + (q_0 + q_+) + 2(r - k + r_* - k_* + A), \qquad (3.85) \\ K_2 &= \lambda - (q_0 + q_+) + 2(k - A) - 2(r_* - k_*). \qquad (3.86) \end{aligned}$$

These two quantities yield (3.42) and (3.43) and complete the proof of Theorem 3.21. ∎

The following consequences of Theorem 3.21 will be used later in our proof.

Corollary 3.23. (**Maximal index jump**) *Let c be a closed geodesic with mean index $\hat{i}(c) > 0$ on a compact simply connected Finsler manifold (M, F) of dimension $d \ge 2$. Denote the basic normal form decomposition of the linearized Poincaré map P_c of c by (3.5). Suppose the integer A in Theorem 3.21 can be chosen to be equal to k given by (3.5). Then there exist infinitely many integers $T \in \mathbf{N}$ such that*

$$\begin{aligned} i(c^{T+1}) - i(c^T) &= \lambda + (q_0 + q_+) + 2r + 2(r_* - k_*) \\ &= i(c) + p_- + p_0 + q_0 + q_+ + r + 2(r_* - k_*). \qquad (3.87) \end{aligned}$$

Proof. If we change the equalities to inequalities so that the right hand side of (3.87) becomes a lower bound of $i(c^{T+1}) - i(c^T)$, then it follows from (3.42) of Theorem 3.21 with $A = k$ immediately.

To get the equality, we choose $A = k$ in Definition 3.15 and (3.40). Together with (3.55)-(3.57), with T chosen by Theorem 3.21 we obtain

$$\begin{aligned} i(c^{T+1}) - i(c^T) &= \lambda + 2\rho(1) + 2\sum_{j=1}^{r}(\mathcal{E}_j^+(T,1) - \mathcal{E}_j(T)) \\ &\quad + 2\sum_{j=k_*+1}^{r_*}(\varphi_j^+(T,1) - \varphi_j(T)) + (q_0 + q_+) \\ &= \lambda + 2k + 2(r - k) + 2(r_* - k_*) + (q_0 + q_+), \end{aligned}$$

which yields (3.87) and completes the proof. ∎

Corollary 3.24. *Let c be a completely non-degenerate closed geodesic with mean index $\hat{i}(c) > 0$ on a compact simply connected Finsler manifold (M, F) of dimension $d \geq 2$. Denote the basic normal form decomposition of the linearized Poincaré map P_c of c by (3.5). Let $r = k$ be the total number of rotation matrices as in (3.5). Then there exists an integer A with $[(r+1)/2] \leq A \leq r$ and infinitely many integers $T \in \mathbf{N}$ such that*

$$i(c^m) - i(c^T) \geq i(c) + (2A - r), \qquad \forall\, m \geq T + 1, \tag{3.88}$$

$$i(c^T) - i(c^m) \geq i(c) - (2A - r), \qquad \forall\, 1 \leq m \leq T - 1. \tag{3.89}$$

Proof. Because c is completely non-degenerate, we have $p_- = p_0 = p_+ = q_- = q_0 = q_+ = 0$, $\lambda = i(c) - r$, $r = k$ and $r_* = k_*$. By Theorem 3.21, we obtain (3.42) and (3.43) for some integer A with $[(r+1)/2] \leq A \leq r$ and some $T \in n\mathbf{N}$, where the constants K_1 and K_2 in (3.42) and (3.43) are given by

$$K_1 = \lambda + 2(r - k) + 2(r_* - k_*) + 2A = i(c) + (2A - r), \tag{3.90}$$

$$K_2 = \lambda + 2k - 2(r_* - k_*) - 2A = i(c) - (2A - r). \tag{3.91}$$

Therefore (3.42) and (3.43) yield (3.88) and (3.89) respectively. ∎

Remark 3.25. Note that Theorem 3.21 and all corollaries hold as well for every symplectic path $\gamma \in \mathcal{P}_\tau(2d)$ by our proofs above. In addition, note that we can choose the T to be some multiple of n in all above properties of Morse indices.

4 Rational and completely non-degenerate closed geodesics

Let (M, F) be a compact manifold with an irreversible or reversible Finsler (including Riemannian) metric F. In this section, we study closed geodesics on M. It is well known that if the total number of prime closed geodesics on M with a bumpy metric F is finite, then every prime closed geodesic c must satisfy $\hat{i}(c) > 0$ by Theorem 2 of [BaK1]. By results of [GrM1], [ViS1], and Theorem 2.4 of [Rad1], we are interested in compact simply connected manifolds. We start from some lemmas.

Lemma 4.1. *Suppose that there exists only one prime closed geodesic c on a compact simply connected bumpy Finsler manifold M with $H^*(M; \mathbf{Q}) = T_{d,h+1}(x)$ for some integers $d \geq 2$ and $h \geq 1$. Then we have*

$$\hat{i}(c) > 0, \quad i(c) = d - 1 \quad \textit{and} \quad M_q = b_q, \qquad \forall\, q \in \mathbf{N}_0. \tag{4.1}$$

Proof. It suffices to prove the last two claims in (4.1).

If $i(c) + d$ is even, $(d + 2j - 1) - i(c)$ is odd. Then by Lemma 2.1 there holds $\overline{C}_{d+2j-1}(E, c^m) = 0$ for all $m \in \mathbf{N}$ and $j \in \mathbf{Z}$. And thus all Morse-type numbers satisfy $M_{d+2j-1} = 0$. But the Morse

inequalities and Lemmas 2.5 and 2.6 then imply the contradiction $0 = M_{d-1} \geq b_{d-1} \geq 1$. So $i(c)+d$ must be odd.

Now by Lemma 2.1 again, we obtain $\overline{C}_{d+2j}(E, c^m) = 0$ for all $m \in \mathbf{N}$ and $j \in \mathbf{Z}$, because $(d+2j) - i(c)$ is odd. Thus $M_{d+2j} = 0 = b_{d+2j}$ by Lemmas 2.5 and 2.6 for all $j \in \mathbf{Z}$. Then $M_q = b_q$ for any $q \in \mathbf{N}_0$ follows from the Morse inequalities.

In addition, by Lemmas 2.5 and 2.6, it yields $b_{d-1} = 1$ and $b_j = 0$ for $j \leq d-2$. Thus by $M_{d-1} = b_{d-1} = 1$ and Lemma 2.1, we get $i(c) = d-1$. ∎

Note that in this section, we denote by $\mathbf{Q}^m$ the m times of the module $\mathbf{Q}$ instead of using the notation $m\mathbf{Q}$ in order to make the text clearer.

In this paper, when there is only one prime closed geodesic c on a Finsler manifold (M, F), we denote the corresponding energy levels by $\kappa_m = E(c^m)$ for $m \geq 1$.

Lemma 4.2. (cf. Proposition 4.1 of [LoD1]) *Let (M, F) be a simply connected compact Finsler manifold with $H^*(M, \mathbf{Q}) = T_{d,h+1}(x)$ and possessing only one prime closed geodesic c which is rational. Let $n = n(c)$ be the analytical period of c. Denote by $C_j = H_j(\overline{\Lambda}, \overline{\Lambda}^{\kappa_n}) = \mathbf{Q}^{c_j}$ for all $j \in \mathbf{Z}$. Then there holds*

$$c_j = b_{j-i(c^n)-p(c)} \qquad \forall\, j \in \mathbf{Z}, \tag{4.2}$$

where b_j is the Betti numbers of the free loop space in Lemma 2.6, the constant $p(c)$ is defined by $p(c) = p(P_c)$ via the linearized Poncaré map P_c of c and definition (3.22).

Next we give a slight modification of Theorem 5.2 of [LoD1] to give a new result which is designed for manifolds in above lemmas with some integer $h \geq 2$ and even integer $d \geq 2$. Here only the condition (4.7) below is weakened slightly than that in [LoD1].

Theorem 4.3. *Let (M, F) be a simply connected compact Finsler manifold with $H^*(M, \mathbf{Q}) = T_{d,h+1}(x)$ and satisfying (OR) with the only prime closed geodesic c. Let $n = n(c)$ be the analytical period of c. Denote by*

$$d_j = k_j^{\epsilon(c^n)}(c^n), \qquad \forall j \in \mathbf{Z}. \tag{4.3}$$

Suppose that there exist two integers $\mu \geq -1$ and $p(c) \geq 0$ such that c satisfies the following conditions:

$$i(c^{m+n}) = i(c^n) + i(c^m) + p(c), \qquad \forall\, m \geq 1, \tag{4.4}$$

$$i(c^m) + \nu(c^m) \leq i(c^n) + \mu, \qquad \forall\, 1 \leq m < n, \tag{4.5}$$

$$d_j = 0, \qquad \forall\, j \geq \mu + 2, \tag{4.6}$$

$$H_{i(c^n)+\mu+1}(\overline{\Lambda}, \overline{\Lambda}^{\kappa_n}) = 0. \tag{4.7}$$

Then there exists an integer $\kappa \geq 0$ such that

$$B(d,q)(i(c^n) + p(c)) + (-1)^{i(c^n)+\mu}\kappa = \sum_{j=\mu-p(c)+1}^{i(c^n)+\mu} (-1)^j b_j. \tag{4.8}$$

Proof of Theorem 4.3. We indicate necessary modifications of the proof of Theorem 5.2 of [LoD1] and are very sketchy here.

As in the Step 1 of the proof of Theorem 5.2 of [LoD1], for $j \in \mathbf{Z}$, we denote by

$$U_j = H_j(\overline{\Lambda}^{\kappa_n}, \overline{\Lambda}^0) = \mathbf{Q}^{u_j}, \quad B_j = H_j(\overline{\Lambda}, \overline{\Lambda}^0) = \mathbf{Q}^{b_j}, \quad C_j = H_j(\overline{\Lambda}, \overline{\Lambda}^{\kappa_n}) = \mathbf{Q}^{c_j}. \tag{4.9}$$

Let $\beta = i(c^n)$. Then the long exact sequence of the triple $(\overline{\Lambda}, \overline{\Lambda}^{\kappa_n}, \overline{\Lambda}^0)$ yields the following diagram:

$$\begin{array}{ccccccccccccccc}
C_{\beta+\mu+1} & \to & U_{\beta+\mu} & \to & B_{\beta+\mu} & \to & C_{\beta+\mu} & \to & \cdots & \to & U_0 & \to & B_0 & \to & C_0 \\
\| & & \| & & \| & & \| & & & & \| & & \| & & \| \\
0 & & \mathbf{Q}^{u_{\beta+\mu}} & & \mathbf{Q}^{b_{\beta+\mu}} & & \mathbf{Q}^{c_{\beta+\mu}} & & \cdots & & \mathbf{Q}^{u_0} & & 0 & & 0,
\end{array}$$

where $C_{\beta+\mu+1} = 0 = C_0$ follows from (4.7) and Lemma 4.2, and $b_0 = 0$ follows from Lemma 2.6. Then this long exact sequence yields

$$0 = \sum_{j=0}^{\beta+\mu} (-1)^j (u_j - b_j + c_j). \tag{4.10}$$

Replacing (5.17) in [LoD1] by the above (4.10), repeating the proof of Theorem 5.2 in [LoD1] and using the above Lemma 4.2, we obtain

$$\begin{aligned}
0 &= B(d,q)(\beta + p(c)) - \sum_{j=0}^{\beta+\mu} (-1)^j b_j + \sum_{j=0}^{\beta+\mu} (-1)^j b_{j-\beta-p(c)} + (-1)^{\beta+\mu} u_{\beta+\mu+1} \\
&= B(d,q)(\beta + p(c)) - \sum_{j=\mu-p(c)+1}^{\beta+\mu} (-1)^j b_j + (-1)^{\beta+\mu} u_{\beta+\mu+1}.
\end{aligned} \tag{4.11}$$

That is, (4.8) holds with $\kappa = u_{\beta+\mu+1} \geq 0$. ∎

Our main result in this section generalizes the multiplicity results in [LoD1] on rational closed geodesics on spheres, and in [DuL1] and [Rad4] on bumpy spheres, and [Rad5] on bumpy $\mathbf{CP}^2$ to all compact simply connected manifolds.

Theorem 4.4. *Let M be a compact simply connected manifold with $H^*(M; \mathbf{Q}) \cong T_{d,h+1}(x)$ for some integers $h \geq 1$ and $d \geq 2$. Let F be an irreversible Finsler metric on M and c be the only prime closed geodesic on M. Then c can be neither rational nor completely non-degenerate.*

Proof. Note that when d is odd, then $h = 1$ by Remark 2.5 of [Rad1]. Note that when $h = 1$, M is rationally homotopic to the sphere S^d. In this case the conclusion that c can not be rational follows from [LoD1], the conclusion that c can not be completely non-degenerate follows from [DuL1]. Therefore it suffices to prove the theorem for the integer $h \geq 2$ and even integer $d \geq 2$. We continue the proof in two claims.

Claim 1: *c is not rational.*

In fact, assuming that c is rational, we follow ideas in the proof of Theorem 6.1 of [LoD1] and prove the theorem by contradiction.

To generate the non-trivial $H_{d-1}(\Lambda M/S^1, \Lambda M^0/S^1; \mathbf{Q})$ (cf. Lemmas 2.5 and 2.6), the prime closed geodesic c must satisfy

$$\hat{i}(c) > 0, \quad 0 \le i(c) \le d-1. \tag{4.12}$$

Let $n = n(c)$ be the analytical period of c. By the periodicity property (A) of Theorem 3.7 of [LoD1], we have

$$i(c^{mn}) = m\, i(c^n) + (m-1)p(c), \qquad \forall\, m \in \mathbf{N}. \tag{4.13}$$

Thus by (4.12) and Corollary 9.2.7 of [Lon4] we have $i(c^n) + p(c) = \hat{i}(c^n) = n\hat{i}(c) > 0$. Note that $i(c^n) = p(c)$ mod 2 by (D) of Theorem 3.7 of [LoD1], thus we have

$$i(c^n) + p(c) \in 2\mathbf{N}. \tag{4.14}$$

Let $\mu = p(c) + (dh-3)$. Then by (4.14) we have

$$i(c^n) + \mu \ge dh - 1 \ge 3, \qquad i(c^n) + \mu \in 2\mathbf{N} - 1. \tag{4.15}$$

Now we can verify the conditions (4.4)-(4.7) of Theorem 4.3 as in the proof of Theorem 6.1 of [LoD1]. Note that (4.4)-(4.5) follow from Theorem 3.7 and Proposition 3.11 of [LoD1], (4.6) follows from (B-2) of Theorem 4.1 of [LoD1], and (4.7) follows from Lemmas 2.6 and 4.2 and the evenness of $\mu + 1 - p(c)$. Then by Theorem 4.3, we obtain for some integer $\kappa \ge 0$:

$$B(d,h)(i(c^n) + p(c)) + (-1)^{i(c^n)+\mu}\kappa = \sum_{j=\mu-p(c)+1}^{i(c^n)+\mu} (-1)^j b_j. \tag{4.16}$$

Thus by (4.15) we obtain

$$B(d,h)(i(c^n) + p(c)) \ge -\sum_{\mu-p(c)+1 \le 2j-1 \le i(c^n)+\mu} b_{2j-1}. \tag{4.17}$$

By Lemma 2.4 we have

$$B(d,h) = -\frac{h(h+1)d}{2D} < 0.$$

Thus from Theorem 3.7 of [LoD1], we have

$$i(c^n) + \mu - (d-1) = i(c^n) + p(c) + dh - d - 2 \in 2\mathbf{N}. \tag{4.18}$$

By (4.17), (4.18) and (2.12) we obtain

$$\begin{aligned} i(c^n) + p(c) &\le -\frac{1}{B(d,h)} \sum_{\mu-p(c)+1 \le 2j-1 \le i(c^n)+\mu} b_{2j-1} \\ &= \frac{2D}{h(h+1)d}\left(\sum_{0 \le 2j-1 \le i(c^n)+\mu} b_{2j-1} - \sum_{0 \le 2j-1 \le dh-2} b_{2j-1} \right). \end{aligned} \tag{4.19}$$

Letting $D = d(h+1) - 2$. Note that because $i(c) + p(c) \geq 2$ by (4.14), we have

$$i(c^n) + \mu = i(c^n) + p(c) + dh - 3 \geq d - 1 + (h-1)d. \tag{4.20}$$

Thus by Lemma 2.6 we have

$$\sum_{0 \leq 2j-1 \leq i(c^n)+\mu} b_{2j-1} = \frac{h(h+1)d}{2D}(i(c^n) + \mu - (d-1)) - \frac{h(h-1)d}{4} + 1 + \epsilon_{d,h}(i(c^n) + \mu). \tag{4.21}$$

On the other hand, because $dh - 3 < dh - 1 = d - 1 + (h-1)d$, by Lemma 2.6 we have

$$\begin{aligned} \sum_{0 \leq 2j-1 \leq dh-3} b_{2j-1} &= \sum_{d-1 \leq 2j-1 \leq dh-3} \left([\frac{2j-1-(d-1)}{d}] + 1 \right) \\ &= \sum_{d \leq 2j \leq dh-2} [\frac{2j}{d}] \\ &= \sum_{\frac{d}{2} \leq j \leq \frac{dh}{2}-1} [\frac{j}{d/2}] \\ &= \frac{dh/2 - 1 - (d/2-1)}{d/2} (\frac{dh/2 - 1 - (d/2-1)}{d/2} + 1) \frac{1}{2} \cdot \frac{d}{2} \\ &= \frac{dh(h-1)}{4}. \end{aligned} \tag{4.22}$$

Therefore we get

$$\begin{aligned} &\sum_{0 \leq 2j-1 \leq i(c^n)+\mu} b_{2j-1} - \sum_{0 \leq 2j-1 \leq dh-3} b_{2j-1} \\ &= \frac{h(h+1)d}{2D}(i(c^n) + \mu - (d-1)) - \frac{h(h-1)d}{4} + 1 + \epsilon_{d,h}(i(c^n) + \mu) \\ &\qquad - \sum_{d-1 \leq 2j-1 \leq dh-3} \left([\frac{2j-1-(d-1)}{d}] + 1 \right) \\ &= \frac{h(h+1)d}{2D}(i(c^n) + p(c) + dh - d - 2) - \frac{dh(h-1)}{2} + 1 + \epsilon_{d,h}(i(c^n) + \mu). \end{aligned} \tag{4.23}$$

Then (4.19) becomes

$$i(c^n) + p(c) \leq i(c^n) + p(c) + dh - d - 2 + \frac{2D}{h(h+1)d} \left(1 - \frac{dh(h-1)}{2} + \epsilon_{d,h}(i(c^n) + \mu) \right),$$

that is,

$$\begin{aligned} \epsilon_{d,h}(i(c^n) + \mu) &\geq \frac{h(h+1)d}{2D} \left(d + 2 + \frac{(h-1)D}{h+1} - dh - \frac{2D}{h(h+1)d} \right) \\ &= \frac{dh - (d-2)}{dh + (d-2)}. \end{aligned} \tag{4.24}$$

Note that by (4.18) we have

$$i(c^n) + \mu - (d-1) = i(c^n) + p(c) + dh - d - 2 = i(c^n) + p(c) - 2d + D. \tag{4.25}$$

Let $\eta \in [0, D/2-1]$ be an integer such that

$$\frac{2\eta}{D} = \{\frac{i(c^n)+p(c)-2d}{D}\} = \{\frac{i(c^n)+\mu-(d-1)}{D}\}. \tag{4.26}$$

By the definition (2.13) of $\epsilon_{d,h}(i(c^n)+\mu)$ and (4.26), we obtain

$$\begin{aligned}
\epsilon_{d,h}(i(c^n)+\mu) &= \{\frac{D}{dh}\{\frac{i(c^n)+\mu-(d-1)}{D}\}\} - (\frac{2}{d}+\frac{d-2}{dh})\{\frac{i(c^n)+\mu-(d-1)}{D}\} \\
&\qquad -h\{\frac{D}{2}\{\frac{i(c^n)+\mu-(d-1)}{D}\}\} - \{\frac{D}{d}\{\frac{i(c^n)+\mu-(d-1)}{D}\}\} \\
&= \{\frac{2\eta}{dh}\} - (\frac{2}{d}+\frac{d-2}{dh})\frac{2\eta}{D} - h\{\frac{2\eta}{2}\} - \{\frac{2\eta}{d}\} \\
&= \{\frac{2\eta}{dh}\} - (\frac{2}{d}+\frac{d-2}{dh})\frac{2\eta}{D} - \{\frac{2\eta}{d}\} \\
&\equiv \epsilon(2\eta).
\end{aligned} \tag{4.27}$$

Now we claim

$$\epsilon(2\eta) < \frac{dh-(d-2)}{dh+(d-2)}, \qquad \forall\ 2\eta \in [0, dh-2]. \tag{4.28}$$

In fact, we write

$$2\eta = pd + 2m \qquad \text{with some } \ p \in \mathbf{N}_0, \ \ 2m \in [0, d-2]. \tag{4.29}$$

Then from $pd + 2m = 2\eta \leq dh-2 = (h-1)d + d - 2$ we have

$$p \in [0, h-1]. \tag{4.30}$$

Therefore in this case we obtain

$$\begin{aligned}
\epsilon(2\eta) &= \frac{pd+2m}{dh} - (\frac{2}{d}+\frac{d-2}{dh})\frac{pd+2m}{D} - \frac{2m}{d} \\
&= \frac{p}{h} - \frac{(2h+d-2)p}{hD} + \frac{2m}{dh} - \frac{(2h+d-2)2m}{dhD} - \frac{2m}{d} \\
&= \frac{p}{h}(1-\frac{2h+d-2}{D}) + \frac{2m}{d}(\frac{1}{h} - \frac{2h+d-2}{hD} - 1) \\
&= \frac{p(d-2)-2mh}{D} \\
&\leq \frac{(h-1)(d-2)}{D}.
\end{aligned} \tag{4.31}$$

Now if (4.28) does not hold, we then obtain

$$\frac{dh-(d-2)}{D} \leq \epsilon(2\eta) \leq \frac{(h-1)(d-2)}{D},$$

that is,

$$dh - d + 2 \leq dh - d + 2 - 2h.$$

Because $h \geq 2$, this yields a contradiction and completes the proof of (4.28).

If $d=2$, then $\{\frac{2\eta}{d}\}=0$ holds in (4.28). Thus the definition (4.27) implies

$$\epsilon(2\eta)\le\epsilon(dh-2),\qquad\forall\,2\eta\in[dh,D-2]. \tag{4.32}$$

If $d\ge 4$, for any $2\eta\in[dh,D-2]$, write $2\eta=pdh+2m$ for some $p\in\mathbf{N}_0$ and $2m\in[0,dh-2]$. Then from $D-2=(h+1)d-4=hd+d-4$ we obtain $p\le 1$ and $2m\le d-4$. Thus we have

$$\begin{aligned}\epsilon(2\eta)&=\frac{2m}{dh}-(\frac{2}{d}+\frac{d-2}{dh})\frac{pdh+2m}{D}-\frac{2m}{d}\\&=\epsilon(2m)-(\frac{2}{d}+\frac{d-2}{dh})\frac{pdh}{D}\\&\le\epsilon(2m).\end{aligned} \tag{4.33}$$

Therefore from (4.28), (4.32) and (4.33), we obtain that (4.28) holds in fact for all integer $\eta\in[0,D/2-1]$. This contradicts (4.24) and completes the proof of Claim 1.

Claim 2: *c is not completely non-degenerate.*

In fact, assuming that c is completely non-degenerate, in which case (M,F) becomes bumpy, we prove the theorem by contradiction.

Then by Theorems 3.2 and 3.3, we have the precise index iteration formulae

$$i(c^m)=m(i(c)-r)+2\sum_{j=1}^{r}\left[\frac{m\theta_j}{2\pi}\right]+r,\qquad\text{where }\frac{\theta_j}{2\pi}\in(0,1)\setminus\mathbf{Q},\ 1\le j\le r. \tag{4.34}$$

By Claim 1 and the mean index identity, we have $r\ge 2$. Note that Claim 2 was proved in [DuL1] and [Rad4] when d is odd or $h=1$, and in [Rad5] when $d=h=2$. Next we give the proof of Claim 2 in two cases for all the values of $d\ge 2$ and $h\ge 1$, which yields also a new proof for the results in [DuL1], [Rad4], and [Rad5].

Case 1: *$H^*(M;\mathbf{Q})=T_{d,h+1}(x)$ with $d=2$ and $h\ge 1$.*

In this case, by the index iteration formulae (4.34) and Lemma 4.1, it yields

$$i(c)=d-1=1, \tag{4.35}$$

$$i(c^{2j})=i(c^2)\pmod 2,\quad i(c^{2j-1})=i(c)\pmod 2,\qquad\forall\,j\ge 1. \tag{4.36}$$

By Lemma 2.6, for any odd integer $k\ge 2h+1$ the Betti numbers b_j in this case satisfy

$$\begin{aligned}\sum_{j=0}^{k}b_j&=h(h+1)\frac{k-1}{2h}-\frac{h(h-1)}{2}+1-(h+1)\{h\{\frac{k-1}{2h}\}\}\\&=\frac{(h+1)(k-h+1)}{2},\end{aligned} \tag{4.37}$$

where we have used the fact $\{h\{\frac{2m}{2h}\}\}=0$ for any $m\in\mathbf{Z}$.

Note that, by Theorem 3.21 there exists an integer subset P of $\{1,\ldots,r\}$ containing r_1 integers with $[(r+1)/2] \leq r_1 \leq r-1$, without loss of generality we assume $P=\{1,\ldots,r_1\}$, such that for any given $\epsilon \in (0,1/4)$ there exists a sufficiently large $T \in 2\mathbf{N}$ satisfying

$$1-\{\frac{T\theta_j}{2\pi}\} < \frac{\epsilon}{r}, \qquad \forall\, 1 \leq j \leq r_1, \tag{4.38}$$

$$\{\frac{T\theta_j}{2\pi}\} < \frac{\epsilon}{r}, \qquad \forall\, r_1+1 \leq j \leq r. \tag{4.39}$$

Thus, by Lemma 4.1 and Corollary 3.24 with $A=r_1$, we can choose $T \in 2\mathbf{N}$ sufficiently large such that $R \equiv i(c^T) \geq 2h+1$ and

$$i(c^m)-i(c^T) \;\geq\; i(c)-r+2A = 1+2r_1-r, \qquad \forall\, m \geq T+1, \tag{4.40}$$

$$i(c^T)-i(c^m) \;\geq\; i(c)+r-2A = 1-(2r_1-r), \qquad \forall 1 \leq m \leq T-1. \tag{4.41}$$

Case 1-1: $R \equiv i(c^T) \in 2\mathbf{Z}+1$.

In this subcase, because T is even, r must be odd by (4.34). By Claim 1 and Lemma 2.4 we must have $r \geq 2$. Therefore together with (4.36) we must have

$$i(c^2) \in 2\mathbf{Z}+1, \quad \text{and} \quad 3 \leq r \in 2\mathbf{N}-1. \tag{4.42}$$

Here we have $n=n(c)=1$ in Lemmas 2.4 and 3.10. Then by the facts $B(2,h)=-\frac{h+1}{2}$ and $\hat{i}(c)=1-r+\sum_{j=1}^{r}\frac{\theta_j}{\pi}$ which follows from (4.34), by Lemma 2.4 we get

$$1-r+\sum_{j=1}^{r}\frac{\theta_j}{\pi} = \frac{2}{h+1}. \tag{4.43}$$

Thus by (4.34) we obtain

$$\begin{aligned} R &\equiv i(c^T) \\ &= T(1-r)+2\sum_{j=1}^{r}\left[\frac{T\theta_j}{2\pi}\right]+r \\ &= T(1-r)+2\sum_{j=1}^{r}\frac{T\theta_j}{2\pi}-2\sum_{j=1}^{r}\left\{\frac{T\theta_j}{2\pi}\right\}+r \\ &< T(1-r)+2\sum_{j=1}^{r}\frac{T\theta_j}{2\pi}-(2r_1-r)+2\epsilon \\ &= \frac{2T}{h+1}-(2r_1-r)+2\epsilon, \end{aligned} \tag{4.44}$$

where the first inequality follows from (4.38).

On the other hand, by (4.35), (4.36), (4.42) and Lemma 2.1, every iteration c^m with $m \geq 1$ contributes 1 to the corresponding Morse-type number $M_{i(c^m)}$. Let

$$\tilde{R} = R+2r_1-r-1. \tag{4.45}$$

Note that $\tilde{R} \geq R$ holds and it is odd. Therefore, by (4.40), (4.41) and Lemma 4.1, we have

$$\sum_{j=0}^{\tilde{R}} b_j = \sum_{j=0}^{\tilde{R}} M_j = T. \tag{4.46}$$

By (4.37) and (4.46), we obtain

$$\frac{(h+1)(\tilde{R}-h+1)}{2} = T. \tag{4.47}$$

Now combining (4.44) and (4.47) together, we get

$$\begin{aligned} R + (2r_1 - r) &< \frac{2T}{h+1} + 2\epsilon \\ &= \frac{2}{h+1} \cdot \frac{(h+1)(\tilde{R}-h+1)}{2} + 2\epsilon \\ &= \tilde{R} - h + 1 + 2\epsilon, \end{aligned} \tag{4.48}$$

which implies $h < 1$. Contradiction!

Case 1-2: $R \equiv i(c^T) \in 2\mathbf{Z}$.

In this subcase, by (4.34), Lemma 2.4, and Claim 1, similarly to the Case 1-1 we obtain

$$i(c^2) \in 2\mathbf{Z} \quad \text{and} \quad 2 \leq r \in 2\mathbf{N}. \tag{4.49}$$

Thus $n = n(c) = 2$ in Lemmas 2.4 and 3.10. Thus from $B(2,h) = -\frac{h+1}{2}$ and $\hat{i}(c) = 1 - r + \sum_{j=1}^{r} \frac{\theta_j}{\pi}$, similarly to (4.43), by Lemma 2.4 we obtain

$$1 - r + \sum_{j=1}^{r} \frac{\theta_j}{\pi} = \frac{1}{h+1}. \tag{4.50}$$

Thus similarly to the proof of (4.44), we obtain

$$R + 2r_1 - r < \frac{T}{h+1} + 2\epsilon. \tag{4.51}$$

On the other hand, it follows from (4.34) and Lemma 2.1 that every c^{2m-1} with $m \geq 1$ contributes 1 to the corresponding Morse-type number $M_{i(c^{2m-1})}$ and every c^{2m} with $m \geq 1$ has no contribution to any Morse-type numbers. Let

$$\tilde{R} = R + 2r_1 - r. \tag{4.52}$$

Note that $\tilde{R} \geq R$ holds and it is even. Therefore by (4.40), (4.41), Lemmas 2.1, 2.6 and 4.1, we obtain

$$\sum_{j=0}^{\tilde{R}-1} b_j = \sum_{j=0}^{\tilde{R}} b_j = \sum_{j=0}^{\tilde{R}} M_j = \frac{T}{2}. \tag{4.53}$$

Then by (4.37) and (4.53), we obtain

$$(h+1)(\tilde{R}-h) = T. \tag{4.54}$$

Now from (4.51) and (4.54) we obtain

$$R + 2r_1 - r < \frac{T}{h+1} + 2\epsilon = \frac{(h+1)(\tilde{R}-h)}{h+1} + 2\epsilon = \tilde{R} - h + 2\epsilon, \tag{4.55}$$

which implies $h < 1$. Contradiction!

Case 2: $H^*(M;\mathbf{Q}) = T_{d,h+1}(x)$ *with even* $d \geq 3$ *and* $h \geq 1$.

In this case, we have $i(c) = d-1$ by Lemma 4.1. By Corollary 3.24, as in the proof of Case 1, we can choose sufficiently large $T \in 2n\mathbf{N}$ with $n = n(c)$ being the analytical period of c such that

$$R \equiv i(c^T) \;\geq\; 2(d-1) + d(h-1) + 1, \tag{4.56}$$

$$i(c^m) - i(c^T) \;\geq\; d - 1 + 2r_1 - r, \qquad \forall\, m \geq T+1, \tag{4.57}$$

$$i(c^T) - i(c^m) \;\geq\; d - 1 - (2r_1 - r), \qquad \forall\, 1 \leq m \leq T-1. \tag{4.58}$$

Let $\tilde{R} = R + 2r_1 - r - (d-1)$. Then it follows from (4.57) and (4.58) that

$$i(c^m) \;\geq\; \tilde{R} + 2(d-1) \geq \tilde{R} + 4, \qquad \forall\, m \geq T+1, \tag{4.59}$$

$$i(c^m) \;\leq\; \tilde{R}, \qquad \forall\, 1 \leq m \leq T-1. \tag{4.60}$$

If $d \geq 4$ and $h \geq 1$, by (4.59) and (4.60) we obtain

$$\{\tilde{R}+1, \ldots, \tilde{R}+5\} \cap \{i(c^m) \,|\, m \geq 1\} = \{\tilde{R}+1, \ldots, \tilde{R}+5\} \cap \{i(c^T)\}.$$

Here note that $R = i(c^T)$ may also miss all of $\tilde{R}+1, \ldots, \tilde{R}+5$. Therefore every c^m with $m \in \mathbf{N}\backslash\{T\}$ has no contribution to the Morse-type numbers $M_{\tilde{R}+1}, \ldots, M_{\tilde{R}+5}$. Note that by (4.56), we have $\tilde{R} > d - 1 + d(h-1)$. Thus by Lemmas 2.5, 2.6 and 4.1 there holds

$$2 \leq \sum_{j=1}^{5} b_{\tilde{R}+j} = \sum_{j=1}^{5} M_{\tilde{R}+j} \leq 1. \tag{4.61}$$

Contradiction!

If $d = 3$, we then have $h = 1$. By (4.59) and (4.60) we obtain

$$\{\tilde{R}+1, \tilde{R}+2, \tilde{R}+3\} \cap \{i(c^m) \,|\, m \geq 1\} = \{\tilde{R}+1, \tilde{R}+2, \tilde{R}+3\} \cap \{i(c^T)\}.$$

Note that in this case, R and r have the same parity by the choice of T, and thus $\tilde{R}$ is even. Similarly to (4.61) by Lemmas 2.5 and 4.1 we then obtain

$$2 = b_{\tilde{R}+2} = \sum_{j=1}^{3} b_{\tilde{R}+j} = \sum_{j=1}^{3} M_{\tilde{R}+j} \leq 1. \tag{4.62}$$

Contradiction!

This completes the proof of Claim 2 and Theorem 4.4. ∎

5 On 4-dimensional compact simply connected irreversible Finsler manifolds

In this section, we give the proof of the main Theorem 1.2 about closed geodesics on 4-dimensional compact simply connected irreversible Finsler manifolds.

By our discussion in Section 1 and Theorems A and B, it suffices to consider the case of the 4-dimensional compact simply connected manifold M satisfying $H^*(M;\mathbf{Q}) \cong T_{d,h+1}(x)$ for some integers $d \geq 2$ and $h \geq 1$ with $hd = 4$. Thus we consider only the following two cases:

$$d = 4 \text{ and } h = 1, \qquad \text{or} \qquad d = h = 2. \tag{5.1}$$

In these two cases, by Lemma 2.4 we have correspondingly

$$B(4,1) = -\frac{2}{3}, \qquad B(2,2) = -\frac{3}{2}. \tag{5.2}$$

Suppose F is an irreversible Finsler metric on M. Assume that c is the only prime closed geodesic on (M, F), and we prove Theorem 1.2 by contradiction.

Denote the basic normal form decomposition of the linearized Poincaré map P_c of c by (3.5). By Theorem 4.4, the closed geodesic c can be neither rational nor completely non-degenerate. Because $\dim M = 4$, together with Rademacher's identity (Lemma 2.4), the basic normal form decomposition of P_c must contain precisely two rotation matrices $R(\theta_j)$ with $\theta_j/(2\pi) \in (0,1) \setminus \mathbf{Q}$ for $j = 1$ and 2, and have the following form:

$$P_c \approx R(\theta_1) \diamond R(\theta_2) \diamond G, \tag{5.3}$$

where G is one of the 2×2 matrices listed below:

$$\begin{cases} N_1(1,a) & \text{with } a = -1, \ 0 \text{ or } 1, \\ N_1(-1,b) & \text{with } b = -1, \ 0 \text{ or } 1, \\ R(\theta_3) & \text{with } \frac{\theta_3}{2\pi} \in ((0,1) \cap \mathbf{Q}) \setminus \{\frac{1}{2}\}. \end{cases} \tag{5.4}$$

Specially in (3.5) of P_c we have

$$k = 2. \tag{5.5}$$

Note that by Lemma 2.4, the irrational numbers

$$\sigma_j = \frac{\theta_j}{2\pi} \tag{5.6}$$

for $j = 1$ and 2 are always linearly dependent on $\mathbf{Q}$ in the following. The following lemma studies the situation in more details.

Lemma 5.1. *Suppose $\sigma_j \in (0,1) \setminus \mathbf{Q}$ for $j = 1$ and 2 satisfy*

$$\sigma_1 + \sigma_2 = \frac{q}{p}, \tag{5.7}$$

for some $p, q \in \mathbf{N}$ and $(p, q) = 1$. Then for any $m \in \mathbf{N}$ there holds

$$[m\sigma_1] + [m\sigma_2] = \begin{cases} [\frac{mq}{p}], & \textit{if} \;\; \{m\sigma_1\} < \{\frac{mq}{p}\}, \\ [\frac{mq}{p}] - 1, & \textit{if} \;\; \{m\sigma_1\} > \{\frac{mq}{p}\}. \end{cases} \tag{5.8}$$

Specially there holds

$$[m\sigma_1] + [m\sigma_2] = [\frac{mq}{p}] - 1, \qquad \textit{when} \quad m \in p\mathbf{N}. \tag{5.9}$$

Proof. Note first that for $m \in \mathbf{N}$ we have

$$\begin{aligned} \{m\sigma_1\} + \{m\sigma_2\} &= \{m\sigma_1\} + \{\frac{mq}{p} - m\sigma_1\} \\ &= \{m\sigma_1\} + \{\{\frac{mq}{p}\} - \{m\sigma_1\}\} \\ &= \begin{cases} \{\frac{mq}{p}\}, & \textit{if} \;\; \{m\sigma_1\} < \{\frac{mq}{p}\}, \\ \{\frac{mq}{p}\} + 1, & \textit{if} \;\; \{m\sigma_1\} > \{\frac{mq}{p}\}. \end{cases} \end{aligned} \tag{5.10}$$

Thus we get

$$\begin{aligned} [m\sigma_1] + [m\sigma_2] &= m(\sigma_1 + \sigma_2) - (\{m\sigma_1\} + \{m\sigma_2\}) \\ &= \frac{mq}{p} - (\{m\sigma_1\} + \{m\sigma_2\}) \\ &= [\frac{mq}{p}] + \{\frac{mq}{p}\} - (\{m\sigma_1\} + \{m\sigma_2\}). \end{aligned}$$

Together with (5.10), it yields (5.8).

When $m \in p\mathbf{N}$, we have always $\{m\sigma_1\} > 0 = \{mq/p\}$ by the irrationality of σ_1. This completes the proof. ∎

We continue the proof of Theorem 1.2 in several steps according to the value of $i(c)$ and the form of G.

Step 1. $i(c) = 0$.

By Theorems 8.1.4-8.1.7 of [Lon3] (cf. Theorem 3.3 above), in this case we must have $G = N_1(1, -1)$, because all the other choices of G in (5.4) yield an odd $i(c)$ by Proposition 3.4. Thus by Theorem 3.3, we have the precise index formulae

$$i(c^m) = -2m + 2([m\sigma_1] + [m\sigma_2]) + 2, \quad \text{and} \quad \nu(c^m) = 1, \quad \forall\, m \in \mathbf{N}. \tag{5.11}$$

Then we have $i(c^m) \in 2\mathbf{Z}$ for all $m \geq 1$ and $n = n(c) = 1$. Thus (5.2) and Lemma 2.4 yield

$$-\frac{1}{|B(d, h)|}(k_0(c) - k_1^+(c)) = \hat{i}(c) = -2 + 2(\sigma_1 + \sigma_2) > 0, \tag{5.12}$$

Then Lemma 2.2, (5.2) and (5.12) imply

$$k_1^+(c^m) = k_1^+(c) = 1, \quad k_0(c^m) = 0, \quad \forall\, m \geq 1. \tag{5.13}$$

Therefore by (5.11), (5.13) and the Morse inequality, we obtain

$$M_{2j} = 0, \qquad b_{2j+1} = M_{2j+1} = {}^{\#}\{m \in \mathbf{N} : i(c^m) = 2j\}, \quad \forall\, j \in \mathbf{N}_0.$$

Specially we have $b_1 = M_1 > 0$. Then we must have

$$d = h = 2, \qquad \text{and} \qquad B(d,h) = -\frac{3}{2}. \tag{5.14}$$

Thus by Lemma 2.6 with $d = h = 2$, we obtain

$$M_{2j} = b_{2j} = 0,\ M_1 = b_1 = 1,\ M_3 = b_3 = 2,\ M_{2j+5} = b_{2j+5} = 3, \qquad \forall\, j \in \mathbf{N}_0. \tag{5.15}$$

Next we estimate $i(c^m)$ using Lemma 5.1. From (5.12)-(5.14) we obtain

$$\sigma_1 + \sigma_2 = \frac{4}{3}. \tag{5.16}$$

Then by Lemma 5.1 we obtain

$$[\frac{4m}{3}] - 1 \le [m\sigma_1] + [m\sigma_2] \le [\frac{4m}{3}], \qquad \forall\, m \in \mathbf{N}. \tag{5.17}$$

Thus by (5.9) and (5.11) for $m = 3k \in \mathbf{N}$ we obtain

$$i(c^{3k}) = -6k + 2([3k \cdot \frac{4}{3}] - 1) + 2 = 2k, \quad \forall\, k \in \mathbf{N}. \tag{5.18}$$

By (5.11) and (5.17) for $m = 3k+1 \in \mathbf{N}$, we obtain

$$-2(3k+1) + 2([(3k+1)\frac{4}{3}] - 1) + 2 \le i(c^{3k+1}) \le -2(3k+1) + 2[(3k+1)\frac{4}{3}] + 2.$$

That is,

$$2k \le i(c^{3k+1}) \le 2k+2, \qquad \forall\, k \in \mathbf{N}_0. \tag{5.19}$$

Similarly for $m = 3k+2 \in \mathbf{N}$, we obtain

$$-2(3k+2) + 2([(3k+2)\frac{4}{3}] - 1) + 2 \le i(c^{3k+2}) \le -2(3k+2) + 2[(3k+2)\frac{4}{3}] + 2.$$

It yields also

$$2k \le i(c^{3k+2}) \le 2k+2, \qquad \forall\, k \in \mathbf{N}_0. \tag{5.20}$$

Now we have the following

Claim 1: *Besides $i(c) = 0$, there hold*

$$i(c^2) = i(c^3) = 2, \tag{5.21}$$

$$i(c^{3m+1}) = i(c^{3m+2}) = i(c^{3m+3}) = 2m+2, \qquad \forall\, m \in \mathbf{N}. \tag{5.22}$$

In fact, by $i(c) = 0$, (5.11), (5.13) and $b_1 = 1$, we obtain

$$i(c^m) \in 2\mathbf{N}, \qquad \forall\, m \ge 2. \tag{5.23}$$

Thus by (5.23), (5.20), (5.18), (5.13) and (5.15), we obtain (5.21).

Now by (5.13) and (5.15), from (5.18)-(5.20) we obtain (5.22) for $m = 1$. Then by an induction argument on m we get (5.22) for all $m \in \mathbf{N}$ and complete the proof of Claim 1.

Now from (5.11), (5.16) and (5.22), for any $m \in \mathbf{N}$ we obtain

$$\begin{aligned} 2m + 2 &= i(c^{3m+1}) \\ &= -2(3m+1) + 2([(3m+1)\sigma_1] + [(3m+1)\sigma_2]) + 2 \\ &= -6m + 2((3m+1)(\sigma_1 + \sigma_2)) - 2(\{(3m+1)\sigma_1\} + \{(3m+1)\sigma_2\}) \\ &= 2m + \frac{8}{3} - 2(\{(3m+1)\sigma_1\} + \{(3m+1)\sigma_2\}). \end{aligned}$$

That is

$$\{(3m+1)\sigma_1\} + \{(3m+1)\sigma_2\} = \frac{1}{3}, \qquad \forall\, m \in \mathbf{N}. \tag{5.24}$$

Similarly to the proof of Lemma 5.1, by (5.16) again for all $m \in \mathbf{N}$ we obtain

$$\begin{aligned} \{(3m+1)\sigma_1\} + \{(3m+1)\sigma_2\} &= \{(3m+1)\sigma_1\} + \{(3m+1)(\frac{4}{3} - \sigma_1)\} \\ &= \{(3m+1)\sigma_1\} + \{\frac{1}{3} - (3m+1)\sigma_1\} \\ &= \{(3m+1)\sigma_1\} + \{\frac{1}{3} - \{(3m+1)\sigma_1\}\}. \end{aligned} \tag{5.25}$$

Because σ_1 is irrational, by a result of A. Granville and Z. Rudnick (cf. the final remark on page 6. of [GrR1]), the sequence $\{(3m+1)\sigma_1\}$ for $m \in \mathbf{N}$ is uniformly distributed mod one on $[0, 1]$. Thus we can find some sufficiently large $m \in \mathbf{N}$ such that

$$\frac{1}{3} < \{(3m+1)\sigma_1\} < 1. \tag{5.26}$$

Plugging it into (5.25) yields the following identity for this m:

$$\{(3m+1)\sigma_1\} + \{(3m+1)\sigma_2\} = \{(3m+1)\sigma_1\} + 1 + \frac{1}{3} - \{(3m+1)\sigma_1\} = \frac{4}{3}, \tag{5.27}$$

which contradicts (5.24). This proves that the Case of $i(c) = 0$ can not happen.

Step 2. $i(c) = 1$.

In this case, by $i(c) = 1$ and Proposition 3.4, the matrix G in (5.3) must be one of the the following matrices:

$$N_1(1, a), \quad N_1(-1, b), \quad \textit{or} \quad R(\theta_3), \tag{5.28}$$

where $a = 0$ or 1, $b = \pm 1$, and $\frac{\theta_3}{2\pi} \in (0, 1) \cap \mathbf{Q}$.

Next we continue our proof in three subcases according to the particular form of the matrix G.

Case 2-1: $G = N_1(-1, -1)$ *or* $R(\theta_3)$ *with* $\frac{\theta_3}{2\pi} \in (0, 1) \cap \mathbf{Q}$.

In this case, the index iteration formulae of $G = N_1(-1, -1)$ and $R(\theta_3)$ are the same, and only their nullities are different. Thus we can use the index formula for $G = R(\theta_3)$ to cover all these

two subcases. As before, we write $\sigma_j = \theta_j/(2\pi)$ for $j = 1, 2, 3$. Then by Theorem 3.3, for $m \geq 1$ we have

$$i(c^m) = -2m + 2\sum_{j=1}^{3} E(m\sigma_j) - 3, \tag{5.29}$$

$$\nu(c^m) = \frac{1+(-1)^m}{2}(q_+ + 2q_0) + 2(r-2) - 2\varphi(m\sigma_3). \tag{5.30}$$

Specially in this case, we have $n = n(c) \geq 2$ and

$$\left\{\begin{array}{c} 1 \leq i(c^m) \in 2\mathbf{N} - 1, \qquad \forall\, m \in \mathbf{N}, \\ \nu(c) = 0, \\ \nu(c^m) = 0, \qquad \text{if } \frac{m}{n} \notin \mathbf{Z}, \\ \nu(c^{mn}) = 2, \qquad \text{for } m \in \mathbf{N}, \qquad \text{if } G = R(\theta_3), \\ \nu(c^{mn}) = 1, \qquad \text{for } m \in \mathbf{N}, \qquad \text{if } G = N_1(-1,-1). \end{array}\right. \tag{5.31}$$

Thus we have

$$k_0(c^m) = k_0(c) = 1, \qquad \text{for } 1 \leq m \leq n-1. \tag{5.32}$$

Next we distinguish two subcases of $d = 4$ with $h = 1$ and $d = h = 2$.

Subcase 2-1-1. *$d = 4$ and $h = 1$.*

In this case, the manifold is rationally homotopic to S^4. We have

Claim 2: *$i(c^n) = 1$, $k_1^+(c^{nm}) = k_1^+(c^n) \equiv k_1 \geq 1$ and $k_0(c^{nm}) = k_0(c^n) = k_2^+(c^{nm}) = k_2^+(c^n) = 0$ for all $m \in \mathbf{N}$.*

In fact, assume $i(c^n) \geq 3$. Then $i(c^{mn}) \geq i(c^n) \geq 3$ for all $m \geq 1$. Together with $i(c) = 1$ and $\nu(c) = 0$, it yields that the Morse type numbers satisfy $M_2 = M_0 = 0$ and $M_1 \geq 1$. Then by Lemma 2.5 with $d = 4$ the Morse inequality yields a contradiction $-1 \geq M_2 - M_1 + M_0 \geq b_2 - b_1 + b_0 = 0$. So $i(c^n) = 1$ must hold.

Assume $k_1^+(c^n) = 0$, by $i(c^m) \in 2\mathbf{N} - 1$ and $\nu(c^m) \leq 2$, we obtain $M_0 = 0$, $M_1 \geq 1$ and

$$M_{2j} = {}^{\#}\{m \in \mathbf{N} \,|\, i(c^{mn}) = 2j-1\}\, k_1^+(c^n) = 0, \qquad \forall\, j \geq 1. \tag{5.33}$$

Then the Morse inequality yields a contradiction $-1 \geq M_2 - M_1 + M_0 \geq b_2 - b_1 + b_0 = 0$. So $k_1^+(c^n) \geq 1$ must hold, and then $k_0(c^n) = k_2^+(c^n) = 0$ by Lemma 2.2. Then by Lemma 2.2 again we get Claim 2 for all $m \in \mathbf{N}$.

In this case, for numbers in the basic normal form decomposition (3.5) of P_c we have $r_* = p_- = p_0 = 0$, $k = 2$, $r + q_0 + q_+ = 3$. Note that Lemma 2.4 yields the linear dependency of $1, \sigma_1, \sigma_2$ over $\mathbf{Q}$. Therefore in Theorem 3.21 we must have $A = 1$, and we can find sufficiently large $T \in n\mathbf{N}$ such that

$$R \equiv i(c^T) \geq 3, \tag{5.34}$$

$$i(c^m) \geq R + 2, \qquad \forall\, m \geq T+1, \tag{5.35}$$

$$i(c^m) \leq R, \qquad \forall\, 1 \leq m \leq T. \tag{5.36}$$

Because all $i(c^m)$ are odd, by (5.35), (5.36), Claim 2 and Lemma 2.1 we obtain that c^ms with $m \geq T+1$ have no contributions to M_js with $0 \leq j \leq R+1$, and c^ms with $1 \leq m \leq T$ have only contributions to M_js with $0 \leq j \leq R+1$. More precisely, $\sum_{2j-1=1}^{R} M_{2j-1}$ is completely contributed by c^ms with all integer $m \leq T$ which are not in $n\mathbf{N}$, and each c^m contributes a 1. And $\sum_{2j=0}^{R+1} M_{2j}$ is completely contributed by c^{mn}s with $m \in \mathbf{N}$ satisfying $mn \leq T$, and each c^{mn} contributes a k_1. Thus we have

$$\begin{aligned} \sum_{j=0}^{R+1}(-1)^j M_j &= \sum_{2j=0}^{R+1} M_{2j} - \sum_{2j-1=1}^{R} M_{2j-1} \\ &= \frac{T}{n}k_1 - \left(T - \frac{T}{n}\right) \\ &= \frac{k_1-(n-1)}{n}T. \end{aligned} \tag{5.37}$$

On the other hand, by (5.2), Claim 2 and Lemma 2.4, we obtain

$$\frac{k_1-(n-1)}{n\hat{i}(c)} = -\frac{2}{3}. \tag{5.38}$$

Thus by (5.37), (5.38), the Morse inequality and Lemma 2.5, we obtain

$$-\frac{2T}{3}\hat{i}(c) = \sum_{j=0}^{R+1}(-1)^j M_j \geq \sum_{j=0}^{R+1}(-1)^j b_j = -\sum_{2k-1=1}^{R} b_{2k-1} \geq 1 - \frac{2R}{3}. \tag{5.39}$$

It implies

$$2R - 2T\hat{i}(c) \geq 3. \tag{5.40}$$

However, by (5.29) we have

$$\begin{aligned} 2R - 2T\hat{i}(c) &= 2\left(-2T + 2\sum_{j=1}^{3} E(T\sigma_j) - 3\right) - 2T\left(2\sum_{j=1}^{3}\sigma_j - 2\right) \\ &= 4\sum_{j=1}^{2}(E(T\sigma_j) - T\sigma_j) - 6 \\ &\leq 2. \end{aligned} \tag{5.41}$$

It contradicts to (5.40) and then completes the proof in this subcase.

Subcase 2-1-2. *$d = h = 2$.*

In this case, the manifold is rationally homotopic to $\mathbf{CP}^2$. Note that Lemma 2.4 yields the linear dependency of $1, \sigma_1, \sigma_2$ over $\mathbf{Q}$. Therefore in Theorem 3.21 we must have $A = 1$, and there exists some $T \in 3n\mathbf{N}$ such that the odd integer $R \equiv i(c^T)$ satisfies $R \geq 5$ and we have

$$i(c^m) \geq R+2, \qquad \forall\, m \geq T+1, \tag{5.42}$$

$$i(c^m) \leq R, \qquad \forall\, 1 \leq m \leq T. \tag{5.43}$$

Let $k_j \equiv k_j^+(c^n)$ for $j = 0$, 1 or 2. The following Claim 3 is crucial.

Claim 3: $k_2^+(c^{nm}) = k_2 = 0$ *for all* $m \in \mathbf{N}$.

If $\nu(c^n) = 1$, then Claim 3 holds automatically by Lemma 2.2. Next we consider the case of $\nu(c^n) = 2$.

Otherwise, we assume $k_2 = 1$. Then by Lemma 2.2 we have

$$k_2^+(c^{nm}) = k_2 = 1, \quad k_0(c^{nm}) = k_0 = k_1^+(c^{nm}) = k_1 = 0, \quad \forall\, m \in \mathbf{N}. \tag{5.44}$$

Then by (5.2) the identity in Lemma 2.4 becomes

$$\frac{-(n-1) - k_2}{n\hat{i}(c)} = B(2,2) = -\frac{3}{2}.$$

Thus by (5.29) and $k_2 = 1$ it yields

$$\sigma_1 + \sigma_2 + \sigma_3 = \frac{4}{3}. \tag{5.45}$$

Since $i(c^m) \in 2\mathbf{N} - 1$ by (5.29), there holds $M_{2j} = 0$ for all $j \in \mathbf{N}_0$ by (5.44) and Lemma 2.2. Thus together with the Morse inequality, it yields

$$M_{2j} = 0, \quad M_{2j+1} = b_{2j+1}, \quad \forall\, j \in \mathbf{N}_0. \tag{5.46}$$

By (5.44) and Lemma 2.2, we have

$$M_{2j-1} = {}^{\#}\{m \in \mathbf{N} \,|\, i(c^m) = 2j-1, \nu(c^m) = 0\} + {}^{\#}\{m \in \mathbf{N} \,|\, i(c^m) = 2j-3, \nu(c^m) = 2\}. \tag{5.47}$$

Let $N_{R+2} = {}^{\#}\{m \in \mathbf{N} \,|\, i(c^m) = R, \nu(c^m) = 2\}$. Then it follows from (5.46), (5.47) and $b_{R+2} = 3$ by Lemma 2.6 that

$$N_{R+2} \le M_{R+2} = b_{R+2} = 3. \tag{5.48}$$

It follows from (5.42)-(5.44) and (5.47)-(5.48) that

$$\sum_{j=0}^{R} M_j = \sum_{2j-1=1}^{R} M_{2j-1} = T - N_{R+2} \ge T - 3. \tag{5.49}$$

On the other hand, by Lemma 2.6 with $d = h = 2$, specially (4.37), we obtain

$$\sum_{j=0}^{R} b_j = \sum_{2j-1=1}^{R} b_{2j-1} = \frac{3(R-1)}{2}. \tag{5.50}$$

So (5.44) and (5.49)-(5.50) yield

$$\frac{3(R-1)}{2} \ge T - 3. \tag{5.51}$$

By (5.29) and the definition of $T \in n\mathbf{N}$ we obtain

$$R = i(c^T) = -2T + 2\sum_{j=1}^{3} E(T\sigma_j) - 3 = -2T + 2\sum_{j=1}^{3} [T\sigma_j] + 1.$$

Therefore by (5.45) we get

$$\begin{aligned} \frac{3(R-1)}{2} &= -3T + 3\sum_{j=1}^{3}[T\sigma_j] \\ &= -3T + 3T(\sigma_1+\sigma_2+\sigma_3) - 3(\{T\sigma_1\}+\{T\sigma_2\}) \\ &= T - 3(\{T\sigma_1\}+\{T\sigma_2\}) \\ &= T - 3. \end{aligned} \tag{5.52}$$

Here the last equality follows from that $\{T\sigma_1\}+\{T\sigma_2\} \in (0,2)$, R is odd and T is an integer multiple of 3, and then $\{T\sigma_1\}+\{T\sigma_2\}$ must be an integer and then is equal to 1. Then (5.49), (5.50) and (5.52) yield $N_{R+2} = 3$. In other words, by (5.43) and the definition of N_{R+2} there exist two distinct integers T_1 and T_2 with $T_1 < T_2 < T$ such that $i(c^{T_1}) = i(c^{T_2}) = R$ and $\nu(c^{T_1}) = \nu(c^{T_2}) = 2$. Because $\sigma_3 \in (0,1)\cap\mathbf{Q}$, there holds $\sigma_3 = q/p$ with some $q < p \in \mathbf{N}$ with $(p,q) = 1$. Therefore $p \geq 2$ holds. Then we have $T - T_2 \geq 2$ and $T_2 - T_1 \geq 2$, thus

$$T - T_1 \geq 4. \tag{5.53}$$

On the other hand, because $i(c^{T_1}) = i(c^T) = R$, replacing T by T_1 equalities in (5.52) still hold, and then it yields

$$T_1 - 3(\{T_1\sigma_1\}+\{T_1\sigma_2\}) = T - 3(\{T\sigma_1\}+\{T\sigma_2\}) = T - 3.$$

Together with (5.53), it implies that

$$0 < 3(\{T_1\sigma_1\}+\{T_1\sigma_2\}) = 3 + (T_1 - T) \leq 3 - 4 = -1. \tag{5.54}$$

This contradiction proves $k_2 = 0$. Then Claim 3 for $m \in \mathbf{N}$ follows from Lemma 2.2.

Because all $i(c^m)$ are odd, by (5.42), (5.43), Claim 3 and Lemma 2.1 we obtain that c^ms with $m \geq T+1$ have no contributions to M_js with $0 \leq j \leq R+1$, and c^ms with $1 \leq m \leq T$ have only contributions to M_js with $0 \leq j \leq R+1$. More precisely, $\sum_{2j-1=1}^{R} M_{2j-1}$ is completely contributed by c^ms with all integer $m \in [1,T]$ which are not in $n\mathbf{N}$ and each c^m contributes a 1, as well as by c^{mn}s with all integer $m \in [1,T/n]$ and each c^{mn} contributes a k_0. And $\sum_{2j=0}^{R+1} M_{2j}$ is completely contributed by c^{mn}s with $m \in \mathbf{N}$ satisfying $mn \in [n,T]$ and each c^{mn} contributes a k_1. Thus we have

$$\begin{aligned} \sum_{j=0}^{R+1}(-1)^j M_j &= \sum_{2j=0}^{R+1} M_{2j} - \sum_{2j-1=1}^{R} M_{2j-1} \\ &= \frac{T}{n}k_1 - \left(\frac{T}{n}k_0 + T - \frac{T}{n}\right) \\ &= \frac{k_1 - (k_0+n-1)}{n}T. \end{aligned} \tag{5.55}$$

On the other hand, by Claim 3 and Lemma 2.4, we have

$$\frac{k_1-(k_0+n-1)}{n\hat{i}(c)} = -\frac{3}{2}. \tag{5.56}$$

Thus by (5.55), (5.56), the Morse inequality and (4.37), we obtain

$$-\frac{3T}{2}\hat{i}(c) = \sum_{j=0}^{R+1}(-1)^j M_j \geq \sum_{j=0}^{R+1}(-1)^j b_j = -\sum_{2k-1=1}^{R} b_{2k-1} = -\frac{3(R-1)}{2}. \tag{5.57}$$

It implies

$$R-1 \geq T\hat{i}(c). \tag{5.58}$$

But on the other hand, by (5.29) we have

$$\begin{aligned} R-T\hat{i}(c) &= (-2T+2\sum_{j=1}^{3}E(T\sigma_j)-3)-T(-2+2\sum_{j=1}^{3}\sigma_j) \\ &= 2\sum_{j=1}^{2}(E(T\sigma_j)-T\sigma_j)-3 \\ &= 2\sum_{j=1}^{2}(1-\{T\sigma_j\})-3 \\ &< 1, \end{aligned} \tag{5.59}$$

which contradicts to (5.58) and completes the proofs in this subcase and Case 2-1.

Case 2-2: $G = N_1(-1,1)$.

In this case, by $i(c)=1$ and Theorem 3.3, we have the iteration formula

$$i(c^m) = -m+2\sum_{j=1}^{2}E(m\sigma_j)-2, \qquad \nu(c^m) = \frac{1+(-1)^m}{2}, \qquad \forall\, m \geq 1. \tag{5.60}$$

Then we have $n=n(c)=2$ and

$$i(c^m) = m \pmod 2, \qquad \nu(c^{2m-1}) = 0 \ \text{ and } \ \nu(c^{2m}) = 1, \qquad \forall\, m \in \mathbf{N}. \tag{5.61}$$

By Lemma 2.2, it yields $k_0^-(c^{2k}) = k_0^-(c^2) = 0$ for all $k \in \mathbf{N}$. Because the iterates c^{2k-1} with $k \in \mathbf{N}$ contribute only to the odd-th Morse-type numbers, we obtain

$$M_{2k} = {}^{\#}\{m \in \mathbf{N} \,|\, i(c^m) = 2k\} k_0^-(c^2) = 0. \tag{5.62}$$

Together with the Morse inequality, it implies that for any $k \geq 1$,

$$b_{2k-1} = M_{2k-1} = {}^{\#}\{j \in \mathbf{N} \,|\, i(c^{2j}) = 2k-2\} k_1^-(c^2) + {}^{\#}\{j \in \mathbf{N} \,|\, i(c^{2j-1}) = 2k-1\}. \tag{5.63}$$

Note that $b_1 = 1$ when $d=2$, and $b_1=0$ when $d=4$. By the facts $i(c)=1$, $\nu(c)=0$ and (5.63) in this case we must have

$$d = h = 2. \tag{5.64}$$

Therefore (5.15) holds again by Lemma 2.6.

Let $k_1 \equiv k_1^-(c^2) \in \{0,1\}$. Then by Lemma 2.4, we obtain

$$\frac{-1-k_1}{2(2(\sigma_1+\sigma_2)-1)} = \sum_{1\le m\le 2,\, 0\le l\le 1} \frac{(-1)^{i(c^m)+l}k_l^\epsilon(c^m)}{n\hat{i}(c)} = B(2,2) = -\frac{3}{2},$$

which yields

$$\sigma_1+\sigma_2 = \frac{4+k_1}{6}. \tag{5.65}$$

Claim 4. $k_1^-(c^{2m}) = k_1 = 1$ *for all* $m \in \mathbf{N}$.

In fact, assume $k_1 = 0$. Then all $i(c^{2k})$ with $k \ge 1$ have no contribution to the odd-th Morse-type number M_{2j-1} with $j \in \mathbf{N}$. In addition, by (5.65), we obtain $3(\sigma_1+\sigma_2) = 2$. Because both of $3\sigma_1$ and $3\sigma_2$ are irrational, it yields $[3\sigma_1]+[3\sigma_2] = 1$. Thus by (5.60), we obtain $i(c^3) = 1$. Together with $i(c) = 1$, it yields $M_1 \ge 2$. It contradicts to the fact $M_1 = b_1 = 1$ obtained from (5.62), (5.63), (5.64) and Lemma 2.6. By Lemma 2.2, Claim 4 is proved.

Next we estimate $i(c^m)$ using Lemma 5.1. By Claim 4, (5.65) becomes

$$\sigma_1+\sigma_2 = \frac{5}{6}. \tag{5.66}$$

Then by Lemma 5.1 we obtain

$$[\frac{5m}{6}] - 1 \le [m\sigma_1] + [m\sigma_2] \le [\frac{5m}{6}], \qquad \forall\, m \in \mathbf{N}. \tag{5.67}$$

Thus by (5.60) and (5.9) for $m = 6k \in \mathbf{N}$ we obtain

$$i(c^{6k}) = -6k + 2([6k\frac{5}{6}] - 1) + 2 = 4k, \quad \forall\, k \in \mathbf{N}. \tag{5.68}$$

By (5.60) and (5.67) for $m = 6k+1 \in \mathbf{N}$, we obtain

$$-(6k+1) + 2([(6k+1)\frac{5}{6}] - 1) + 2 \le i(c^{6k+1}) \le -(6k+1) + 2[(6k+1)\frac{5}{6}] + 2.$$

That is,

$$4k-1 \le i(c^{6k+1}) \le 4k+1, \qquad \forall\, k \in \mathbf{N}_0. \tag{5.69}$$

Similarly for $m = 6k+2 \in \mathbf{N}$, we obtain

$$-(6k+2) + 2([(6k+2)\frac{5}{6}] - 1) + 2 \le i(c^{6k+2}) \le -(6k+2) + 2[(6k+2)\frac{5}{6}] + 2.$$

It yields

$$4k \le i(c^{6k+2}) \le 4k+2, \qquad \forall\, k \in \mathbf{N}_0. \tag{5.70}$$

For $m = 6k+3 \in \mathbf{N}$, we obtain

$$-(6k+3) + 2([(6k+3)\frac{5}{6}] - 1) + 2 \le i(c^{6k+3}) \le -(6k+3) + 2[(6k+3)\frac{5}{6}] + 2.$$

It yields

$$4k + 1 \leq i(c^{6k+3}) \leq 4k + 3, \qquad \forall\, k \in \mathbf{N}_0. \tag{5.71}$$

For $m = 6k + 4 \in \mathbf{N}$, we obtain

$$-(6k+4) + 2([(6k+4)\frac{5}{6}] - 1) + 2 \leq i(c^{6k+4}) \leq -(6k+4) + 2[(6k+4)\frac{5}{6}] + 2.$$

It yields

$$4k + 2 \leq i(c^{6k+4}) \leq 4k + 4, \qquad \forall\, k \in \mathbf{N}_0. \tag{5.72}$$

For $m = 6k + 5 \in \mathbf{N}$, we obtain

$$-(6k+5) + 2([(6k+5)\frac{5}{6}] - 1) + 2 \leq i(c^{6k+5}) \leq -(6k+5) + 2[(6k+5)\frac{5}{6}] + 2.$$

It yields also

$$4k + 3 \leq i(c^{6k+5}) \leq 4k + 5, \qquad \forall\, k \in \mathbf{N}_0. \tag{5.73}$$

Then using similar arguments in the proof of Claim 1, we have

Claim 5: $i(c) = 1$, $i(c^2) = 2$, $i(c^3) = 3$, $i(c^4) = 4$, $i(c^5) = 5$, $i(c^6) = 4$, $i(c^7) \leq 5$.

In fact, (5.15) is crucial in the following. Note that c contributes a 1 to $M_1 = b_1 = 1$ by the facts $i(c) = 1$ and $\nu(c) = 0$. Thus $i(c^m) \geq 2$ for all $m \geq 2$. Then by Claim 4 and (5.70)-(5.71) with $k = 0$ we obtain $i(c^2) \leq 2$ and $i(c^3) \leq 3$. Thus by (5.61) we obtain $i(c^2) = 2$ and $i(c^3) = 3$.

By (5.68) with $k = 1$ and (5.72)-(5.73) with $k = 0$, we obtain $i(c^6) = 4$, $i(c^4) \leq 4$, $i(c^5) \leq 5$. By Claim 4 and (5.15) we obtain $i(c^4) = 4$ and $i(c^5) = 5$.

Then by (5.69) with $k = 1$ we obtain $i(c^7) \leq 5$. Claim 5 is proved.

Now by Claims 4 and 5, each $i(c^k)$ with $1 \leq k \leq 7$ contributes 1 to the Morse-type numbers $M_1 + M_3 + M_5$. Thus by (5.15), (5.62)-(5.63) we obtain

$$6 = \sum_{j=0}^{5} b_j = \sum_{j=0}^{5} M_j \geq 7. \tag{5.74}$$

Contradiction!

Case 2-3: *$G = N_1(1, a)$ with $a = 0$ or 1.*

In this case, by $i(c) = 1$ and Theorem 3.3 we have the formula

$$i(c^m) = 2\sum_{j=1}^{2}[m\sigma_j] + 1, \qquad \nu(c^m) = 2 - a, \qquad \forall\, m \geq 1. \tag{5.75}$$

Note that all $i(c^m)$ with $m \geq 1$ are odd and non-decreasing in m. Because $b_1 = 1$ when $d = h = 2$, and $b_3 = 1$ when $d = 4$ and $h = 1$, to generate the non-zero Morse-type number $M_1 \geq b_1$ or

$M_3 \geq b_3$, there must hold $k_0(c) + k_2^+(c) = 1$ and $k_1^+(c) = 0$. Thus by the Morse inequality and Lemmas 2.2 and 2.4, it yields

$$M_{2k} = 0, \quad M_{2k-1} = b_{2k-1}, \forall\, k \in \mathbf{N}_0, \tag{5.76}$$

$$\sigma_1 + \sigma_2 = \frac{-1}{2B(d,h)}. \tag{5.77}$$

If $d = 4$ and $h = 1$, we have $B(d,h) = -2/3$ and $b_3 = b_5 = b_7 = 1 < 2 = b_9$ by Lemma 2.5. In order to get $M_3 = b_3 = 1$, by $i(c) = 1$ and Lemma 2.2 it yields $k_2^+(c^m) = k_2^+(c) = 1$ and $k_0(c^m) = k_0(c) = k_1^+(c^m) = k_1^+(c) = 0$ for all $m \geq 1$. Note that by (5.77) we have

$$\sigma_1 + \sigma_2 = \frac{3}{4}.$$

Thus $[4\sigma_1] + [4\sigma_2] = 3 - 1 = 2$ by Lemma 5.1. So we have $i(c^4) = 5$ by (5.75). Thus we obtain

$$4 \leq \sum_{2j-1=3}^{7} M_{2j-1} = \sum_{2j-1=3}^{7} b_{2j-1} = 3,$$

a contradiction.

If $d = h = 2$, we have $B(d,h) = -3/2$ and $b_1 = 1$ by Lemma 2.5. To generate $M_1 = b_1 = 1$, we should have $1 = k_0(c) = k_0(c^m)$ and $0 = k_1^+(c) = k_2^+(c) = k_1^+(c^m) = k_2^+(c^m)$ for all $m \in \mathbf{N}$ by $i(c) = 1$ and Lemma 2.2. Notice that

$$\sigma_1 + \sigma_2 = \frac{1}{3}$$

holds by (5.77). Then $[3\sigma_1] + [3\sigma_2] = 1 - 1 = 0$ by Lemma 5.1. Thus $i(c^3) = 1$ by (5.75). Then by the monotone increasing of the Morse indices $i(c^m)$ in m, we obtain $3 \leq M_1 = b_1 = 1$, a contradiction.

This completes the proof of Step 2 for $i(c) = 1$.

Step 3. $i(c) \geq 2$.

Because $i(c^m) \geq i(c)$ for $m \geq 1$ due to the Bott formula, to generate the non-trivial homology $H_{d-1}(\overline{\Lambda} M, \overline{\Lambda}^0 M; \mathbf{Q})$, then the Morse index of c must satisfy $i(c) \leq d - 1$. Thus we must have $d = 4$ and $h = 1$. In other words, the manifold is rationally homotopic to S^4. We continue the proof in two cases according to the value of $i(c)$.

Case 3-1: $i(c) = 2$.

By Proposition 3.4, we must have $G = N_1(1, -1)$ in (5.4). Thus, by Theorem 3.3, we have

$$i(c^m) = 2([m\sigma_1] + [m\sigma_2]) + 2 \quad \text{and} \quad \nu(c^m) = 1, \qquad \forall\, m \in \mathbf{N}. \tag{5.78}$$

Thus in this case, we have $i(c^m) \in 2\mathbf{Z}$ for all $m \in \mathbf{N}$ and are non-decreasing in m, and then $n = n(c) = 1$. Thus by Lemma 2.4 we have the identity

$$-\frac{3}{2}(k_0(c) - k_1^+(c)) = 2(\sigma_1 + \sigma_2) = \hat{i}(c) > 0, \tag{5.79}$$

which implies $k_1^+(c^m) = k_1^+(c) = 1$ and $k_0(c^m) = 0$ for all $m \geq 1$ by Lemma 2.2. So (5.79) becomes

$$\sigma_1 + \sigma_2 = \frac{3}{4}. \tag{5.80}$$

By (5.9) in Lemma 5.1 we obtain $[4\sigma_1] + [4\sigma_2] = 3 - 1 = 2$, and then

$$i(c^4) = 2([4\sigma_1] + [4\sigma_2]) + 2 = 6. \tag{5.81}$$

Thus by Theorem 3.13 we get

$$i(c^m) \leq 6, \qquad \forall\, m = 1, 2, 3, 4. \tag{5.82}$$

From the above discussion, for all integer $k \geq 0$ we get

$$M_{2k} = {}^{\#}\{m \geq 1 : i(c^m) = 2k)\}\, k_0(c) = 0, \tag{5.83}$$

$$M_{2k+1} = {}^{\#}\{m \geq 1 : i(c^m) = 2k\}. \tag{5.84}$$

Thus we have $M_1 + M_3 + M_5 + M_7 \geq 4$ by (5.82) and (5.84). Thus the Morse inequality and Lemma 2.5 again yield a contradiction:

$$-4 \geq \sum_{q=0}^{8} (-1)^q M_q \geq \sum_{q=0}^{8} (-1)^q b_q = -3. \tag{5.85}$$

Case 3-2: $i(c) \geq 3$.

Note that by Theorem 3.13 it yields $i(c^{m+1}) \geq i(c^m)$ for all $m \geq 1$.

By Lemma 2.4, both σ_1 and σ_2 are linearly dependent over $\mathbf{Q}$. Thus we must have $A = 1$ in Theorem 3.21 and there exists some $T \in 12n\mathbf{N}$ with $n = n(c)$ being the analytical period of c such that

$$i(c^m) - i(c^T) \geq i(c) + p_0 + p_- + (q_0 + q_+) + r - 2 \equiv \xi(c), \qquad \forall\, m \geq T + 1, \tag{5.86}$$

$$i(c^T) - i(c^m) \geq i(c) - r + p_- + p_0 + k - (q_0 + q_+) \geq 0, \qquad \forall\, 1 \leq m \leq T - 1, \tag{5.87}$$

where we used the fact $k = 2$ in Theorem 3.21.

Let $\tau^-(m) = \frac{1-(-1)^m}{2}$ for any $m \in \mathbf{N}$. Note that

$$\nu(c^n) = 2(p_0 + q_0) + 2(r - 2) + p_- + q_+ + p_+ + q_-. \tag{5.88}$$

From $i(c) \geq 3$ and the fact $r - 2 + p_0 + q_0 + p_+ + q_- \leq 1$, we get

$$\xi(c) = i(c) + \nu(c^n) - (r - 2 + p_0 + q_0 + p_+ + q_-) \geq \nu(c^n) + 1 + \tau^-(i(c^T) + \nu(c^n)).$$

Then (5.86) becomes

$$i(c^m) - i(c^T) \geq \nu(c^n) + 1 + \tau^-(i(c^T) + \nu(c^n)), \qquad \forall\, m \geq T + 1. \tag{5.89}$$

Let $R = i(c^T)$, $\bar{\nu} = \nu(c^T) = \nu(c^n)$ and $\tilde{R} \equiv R+\bar{\nu}+\tau^-(R+\bar{\nu}) \in 2\mathbf{Z}$. It follows from (5.87) that all iterations c^m with $1 \le m \le T$ contribute only to the Morse-type numbers M_q for $0 \le q \le R+\bar{\nu}$, and from (5.89) that all the iterations c^m with $m \ge T+1$ do not contribute to these Morse-type numbers M_q with $0 \le q \le R+\bar{\nu}$. Thus it yields

$$\begin{aligned}
\sum_{q=0}^{R+\bar{\nu}}(-1)^q M_q &= \sum_{\substack{0\le q\le R+\bar{\nu}\\ 1\le m\le T}} (-1)^q \dim \overline{C}_q(E, c^m) \\
&= \sum_{m=1}^{T}\left(\sum_{q=0}^{R+\bar{\nu}}(-1)^{i(c^m)+(q-i(c^m))} k_{q-i(c^m)}^{\epsilon(c^m)}(c^m)\right) \\
&= \sum_{m=1}^{T}\left(\sum_{q=0}^{i(c^m)+\nu(c^m)}(-1)^{i(c^m)+(q-i(c^m))} k_{q-i(c^m)}^{\epsilon(c^m)}(c^m)\right) \\
&= \sum_{m=1}^{T}\left(\sum_{l_m=0}^{\nu(c^m)}(-1)^{i(c^m)+l_m} k_{l_m}^{\epsilon(c^m)}(c^m)\right) \\
&= \frac{T}{n}\sum_{\substack{1\le m\le n\\ 0\le l_m\le \nu(c^m)}} (-1)^{i(c^m)+l_m} k_{l_m}^{\epsilon(c^m)}(c^m) \\
&= T\hat{i}(c)B(4,1),
\end{aligned} \tag{5.90}$$

where we used (5.87) and (5.89) in the first equality, Lemma 2.1 in the second one, (i) of Lemma 2.2 in the third and fourth ones, Lemma 2.3 in the fifth one, and Lemma 2.4 in the sixth one.

In this case, by Lemma 2.5 only b_qs with odd $q \ge 3$ are non-zero. By (5.90), the Morse inequality and Lemma 2.5 we obtain

$$T\hat{i}(c)B(4,1) = \sum_{j=0}^{\tilde{R}}(-1)^j M_j \ge \sum_{j=0}^{\tilde{R}}(-1)^j b_j = -\sum_{2q-1=1}^{\tilde{R}-1} b_{2q-1} \ge \frac{5-2\tilde{R}}{3}. \tag{5.91}$$

Here $M_{R+\bar{\nu}+\tau^-(R+\bar{\nu})} = 0$ by (5.87) and (5.89) when $R+\bar{\nu}$ is odd. This fact is used in the first equality in (5.91) when $R+\bar{\nu}$ is odd. Note also that in the first inequality of (5.91), the evenness of $\tilde{R}$ implies the availability of the Morse inequality.

On the other hand, by Lemma 2.4 we obtain

$$\frac{-t}{n(s+2(\sigma_1+\sigma_2)+\frac{q}{p})} = B(4,1) = -\frac{2}{3},$$

for some integers s, t, q and p, where we write $q/p = \theta_3/\pi \in [0,1)\cap\mathbf{Q}$ with $(p,q)=1$ when $q>0$ for the possible term $R(\theta_3)$. From this identity we obtain

$$\sigma_1+\sigma_2 = \frac{3t}{4n} - \frac{s}{2} - \frac{q}{2p} = \frac{b}{4n}$$

for some integer $b > 0$, where we have used the fact $p|n$ which follows from Definition 3.6 of $n = n(c)$. Thus according to the choice of $T \in 12n\mathbf{N}$, by (5.10) we obtain

$$\{T\sigma_1\} + \{T\sigma_2\} = 1. \tag{5.92}$$

Also note that (5.88) yields

$$\nu(c^n) - (r - 2 + p_- + p_0 + q_+ + q_0) = p_0 + q_0 + p_+ + q_- + r - 2 \leq 1. \tag{5.93}$$

By Theorem 3.3 and (5.92)-(5.93) we obtain that the integer on the right hand side of (5.90) satisfies

$$\begin{aligned}
T\hat{i}(c)B(4,1) &= -\frac{2T}{3}\left(i(c) + p_- + p_0 - r + 2\sum_{j=1}^{r}\sigma_j\right) \\
&= -\frac{2}{3}\left(T(i(c) + p_- + p_0 - r) + 2\sum_{j=1}^{r}E\left(T\sigma_j\right) - 2\right) \\
&= -\frac{2}{3}\left(i(c^T) + r + p_- + p_0 + q_+ + q_0 - 2\right) \\
&\leq -\frac{2}{3}(R + \nu(c^n) - 1) \\
&\leq -\frac{2}{3}(\tilde{R} - 2),
\end{aligned} \tag{5.94}$$

where the first equality follows from (3.9), the second equality follows from (5.92) and the fact

$$\sum_{j=1}^{r}T\sigma_j = \sum_{j=1}^{r}([T\sigma_j] + \{T\sigma_j\}) = \sum_{j=1}^{r}E(T\sigma_j) - 2 + \sum_{j=1}^{2}\{T\sigma_j\} = \sum_{j=1}^{r}E(T\sigma_j) - 1,$$

the third equality follows from (3.7) with $m = T \in 2\mathbf{N}$, the first inequality follows from (5.93), and the last inequality follows from the definition of $\tilde{R}$.

Now (5.91) and (5.94) yield a contradiction.

The proof of Theorem 1.2 is complete. ∎

6 On compact simply connected reversible Finsler manifolds

In this section, we study closed geodesics on compact simply connected reversible Finsler manifolds, including Riemannian manifolds, and give the proofs of the main Theorems 1.1 and 1.3 about closed geodesics on 4-dimensional compact simply connected reversible Finsler manifolds.

For any reversible Finsler as well as Riemannian metric F on a compact manifold M, the energy functional E is symmetric on every loop $f \in \Lambda M$ and its inverse curve f^{-1} defined by $f^{-1}(t) = f(1-t)$. Thus these two curves have the same energy $E(f) = E(f^{-1})$ and play the same roles in the variational structure of the energy functional E on ΛM. Specially, the m-th iterates c^m and c^{-m} of a closed geodesic c and its inverse curve c^{-1} have precisely the same Morse indices, nullities, and critical modules. Let $n = n(c)$. So there holds

$$\dim \overline{C}_*(E, c^m) = \dim \overline{C}_*(E, c^{-m}). \tag{6.1}$$

Thus if c is the only geometrically distinct prime closed geodesic on M, then all the Morse type numbers must be even, i.e.,

$$M_j \in 2\mathbf{N}_0, \qquad \forall\, j \in \mathbf{Z}, \tag{6.2}$$

and the identity in Lemma 2.4 becomes

$$2 \sum_{\substack{0 \le l_m \le \nu(c^m) \\ 1 \le m \le n}} (-1)^{i(c^m)+l_m} k_{l_m}^{\epsilon(c^m)}(c^m) = n\hat{i}(c)B(d,h). \tag{6.3}$$

From this consideration we get the following result.

Theorem 6.1. *Theorems 4.3 and 4.4 hold for reversible Finsler (as well as Riemannian) metric on the corresponding manifold (M, F) too. Therefore Theorem 1.1 holds.*

Proof. The current version of Theorem 4.3 works in the reversible Finsler metric case without any changes by the same reason as we have explained in Remark 7.1 of [LoD1]. Note that now the integer κ in (4.8) is even by the above reason.

For the Claim 1 of Theorem 4.4 with a reversible Finsler metric on M, by the same reason, the above proof of Theorem 4.4 works without any change and shows that the only geometrically distinct prime closed geodesic c which can not be rational in the reversible case.

For Theorem 4.4 with a reversible Finsler metric on M and only one geometrically distinct prime closed geodesic c on M which is completely non-degenerate, the above proof of Claim 2 in Theorem 4.4 with minor modifications works too. In fact, Lemma 4.1 and (6.2) yields a much simpler proof, because we get the following contradiction immediately

$$1 = b_{dh-1} = M_{dh-1} \in 2\mathbf{N}_0, \tag{6.4}$$

where $dh = \dim M$. Therefore Claim 2 of Theorem 4.4 holds too in the reversible Finsler metric case.

Thus Theorem 1.1 holds. ∎

Now we can give

The proof of Theorem 1.3. This proof is similar to that of Theorem 1.2 in Section 5. Next we follow the classification used in the proof of Theorem 1.2 and indicate only some necessary changes and omit the details.

Step 1. $i(c) = 0$.

Following the study in Step 1 of the proof of Theorem 1.2, we have $i(c) = 0$ and $G = N_1(-1, 1)$ in (5.4). By (6.1), the positive numbers $1/|B(d,h)|$ should be replaced by $2/|B(d,h)|$ in (5.12). Then similar arguments yield (5.15), specially by (6.2) we obtain the following contradiction

$$1 = b_1 = M_1 \in 2\mathbf{N}_0, \tag{6.5}$$

and then complete the proof in Step 1.

Step 2. $i(c) = 1$.

Case 2-1. As in the proof of Theorem 1.2, we distinguish two subcases.

Subcase 2-1-1. $d = 4$ and $h = 1$.

Replacing $k_1^+(c^n)$ by $2k_1^+(c^n)$ in (5.33), by the same proof we get Claim 2. Then replacing c^m by c^m and c^{-m}, c^{mn} by c^{mn} and c^{-mn} in the paragraph below (5.36), instead of (5.37) and (5.38), by (6.2) and (6.3) we obtain

$$\sum_{j=0}^{R+1}(-1)^j M_j = 2T\frac{k_1 - n + 1}{n}, \tag{6.6}$$

$$2\frac{k_1 - (n-1)}{n\hat{i}(c)} = -\frac{2}{3}. \tag{6.7}$$

Then using (6.6) and (6.7), the same proofs from (5.39) to (5.41) yield a contradiction.

Subcase 2-1-2. $d = h = 2$.

In this subcase, note that we have still (5.46) if $k_2^+(c^n) = 1$. Thus the contradiction $1 = b_1 = M_1 \in 2\mathbf{N}_0$ yields Claim 3.

Now as in the above Subcase 2-1-1, the proofs in (5.55) to (5.59) yield a contradiction.

Case 2-2. Similarly to (5.60)-(5.64), we obtain $1 = b_1 = M_1 \in 2\mathbf{N}_0$, contradiction!

Case 2-3. In this case, from (5.76)-(5.77) and (6.2) we obtain the contradiction $1 = b_{d-1} = M_{d-1} \in 2\mathbf{Z}$ with $d = 2$ or $d = 4$.

Step 3. $i(c) \geq 2$.

Case 3-1. $i(c) = 2$.

In this case $G = N_1(1, -1)$. By (6.3) the identity (5.80) now becomes

$$-3(k_0(c) - k_1^+(c)) = 2(\sigma_1 + \sigma_2) = \hat{i}(c) > 0, \tag{6.8}$$

with $k_0(c) = 0$ and $k_1^+(c) = 1$, and

$$\sigma_1 + \sigma_2 = \frac{3}{2}. \tag{6.9}$$

By (5.79), this specially implies

$$M_{2k} = 0, \qquad M_{2k+1} = b_{2k+1} \qquad \forall\, k \in \mathbf{N}_0.$$

Thus by Lemma 5.1 we obtain $[2\sigma_1] + [2\sigma_2] = 3 - 1 = 2$, and then

$$i(c^2) = 2([2\sigma_1] + [2\sigma_2]) + 2 = 6. \tag{6.10}$$

Thus by the monotone increasing of $i(c^m)$ in m from Theorem 3.13, we obtain the following contradiction

$$0 = M_5 = b_5 \geq 1.$$

Case 3-2. $i(c) \geq 3$.

Because the contributions of c^{-m} with $m \geq 1$, similarly to (5.90) by (6.2) and (6.3) we obtain

$$\begin{aligned}
\sum_{q=0}^{R+\bar{\nu}}(-1)^q M_q &= 2\sum_{\substack{0\leq q\leq R+\bar{\nu}\\ 1\leq m\leq T}}(-1)^q \dim \overline{C}_q(E, c^m) \\
&= \frac{2T}{n}\sum_{\substack{1\leq m\leq n\\ 0\leq l_m\leq \nu(c^m)}}(-1)^{i(c^m)+l_m} k_{l_m}^{\epsilon(c^m)}(c^m) \\
&= T\hat{i}(c)B(4,1). \qquad (6.11)
\end{aligned}$$

Then by the same proof of (5.91) and (5.94) we obtain a contradiction.

The proof of Theorem 1.3 is complete. ∎

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